Abstract Models of Computation and Complexity Lower Bounds

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Discrete logarithm (DL) problem

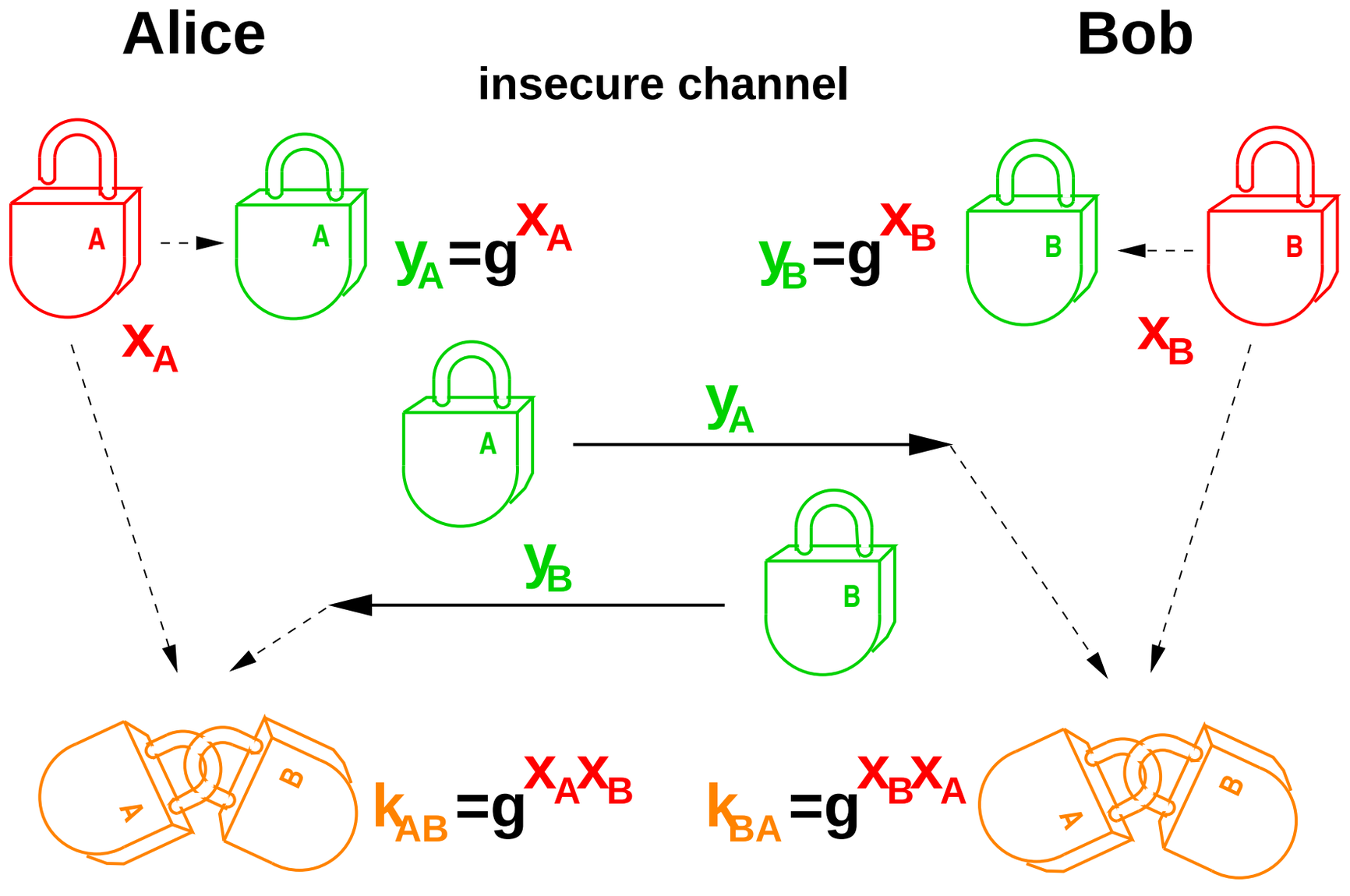
Cyclic group G of order n with generator g:

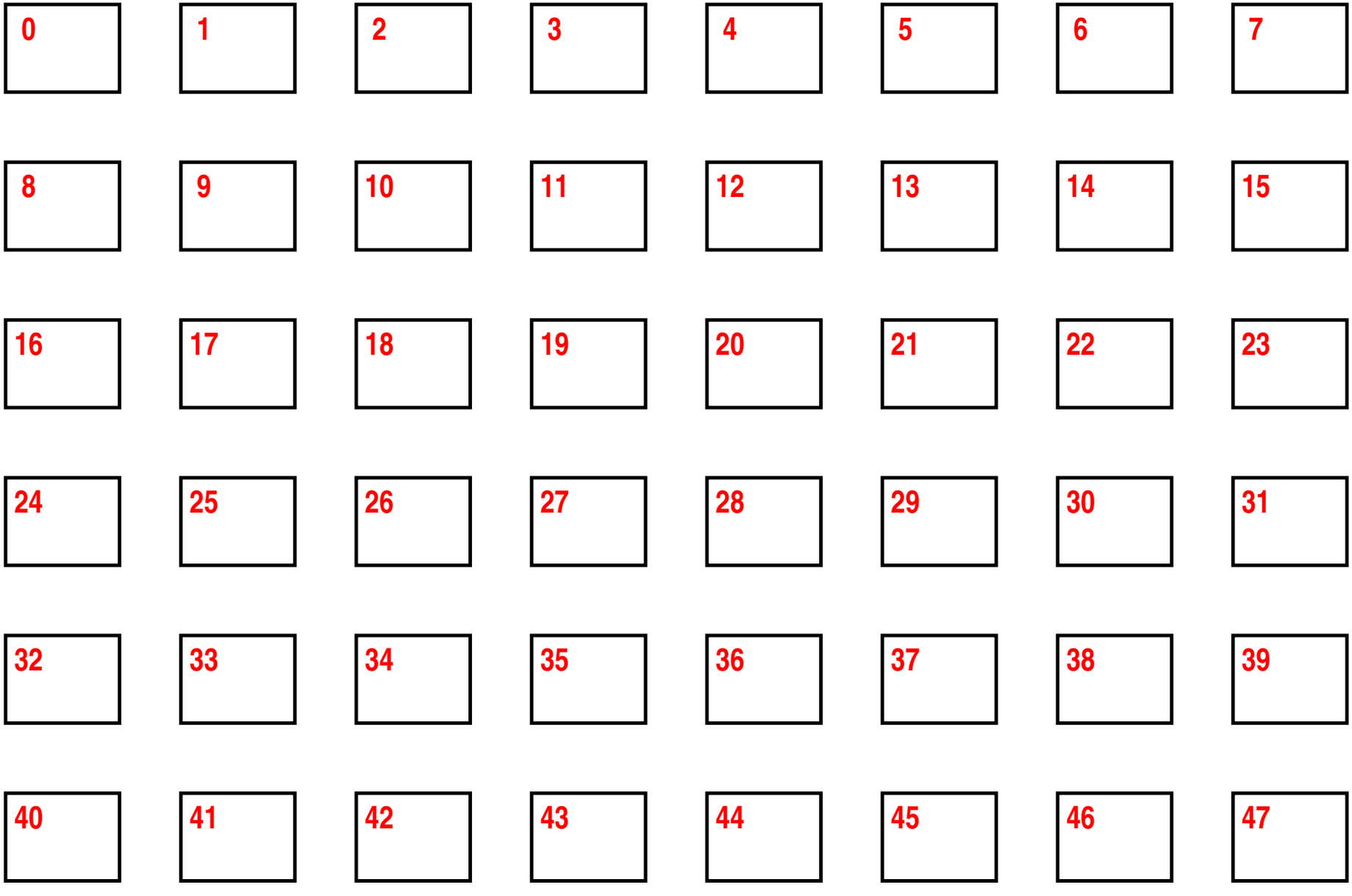
$$\mathbf{G} = \langle \mathbf{g} \rangle = \{\mathbf{g}^i : 0 \leq i < n\}$$

DL problem: Given a $\in$ G, find x such that a $=$ $\mathbf{g}^{\mathbf{x}}$.

$\mathbf{x} \longrightarrow \mathbf{g}^{\mathbf{x}}$ is for many groups believed to be a OWF.

Diffie-Hellman protocol: mechanical analog

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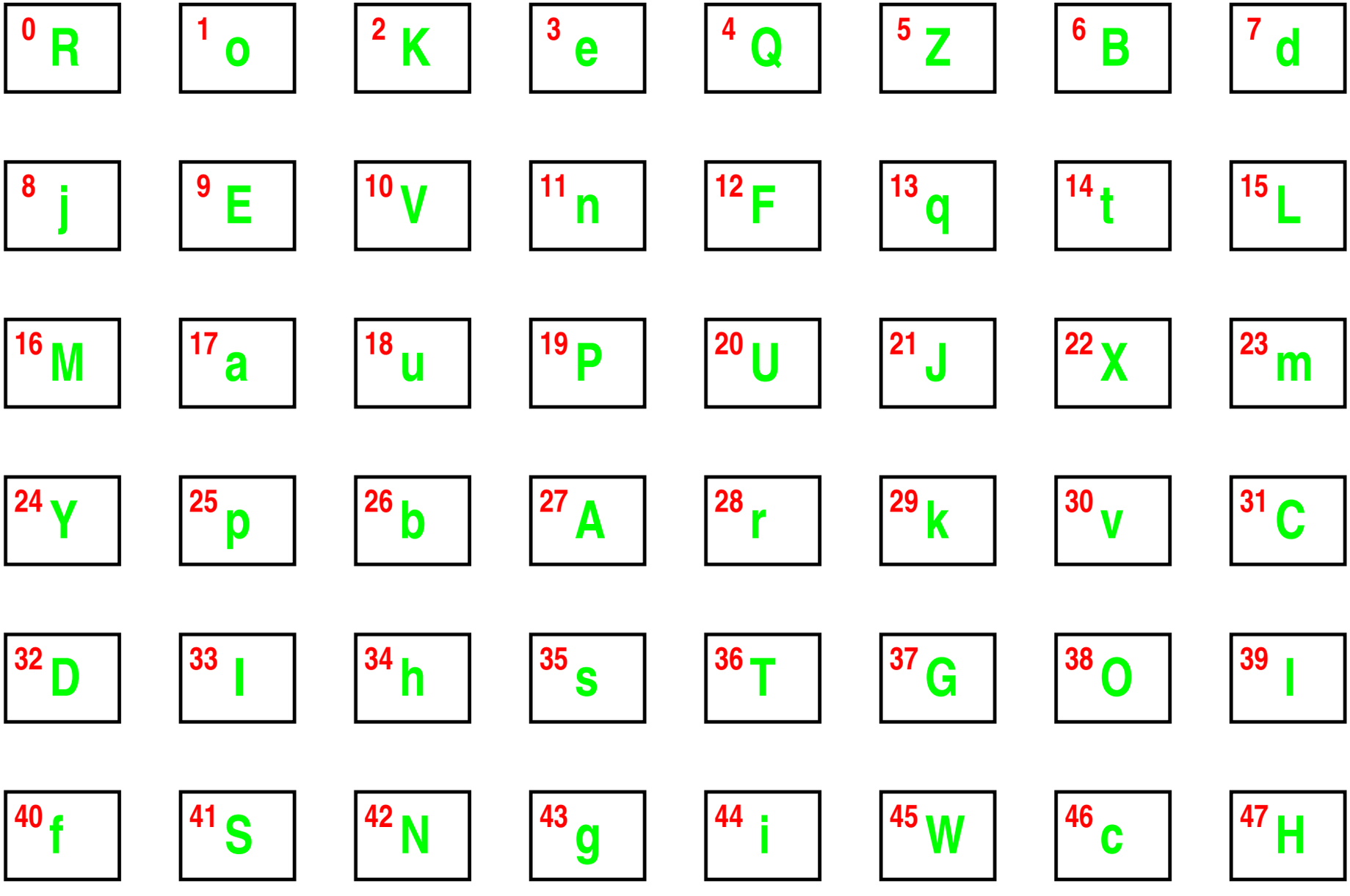
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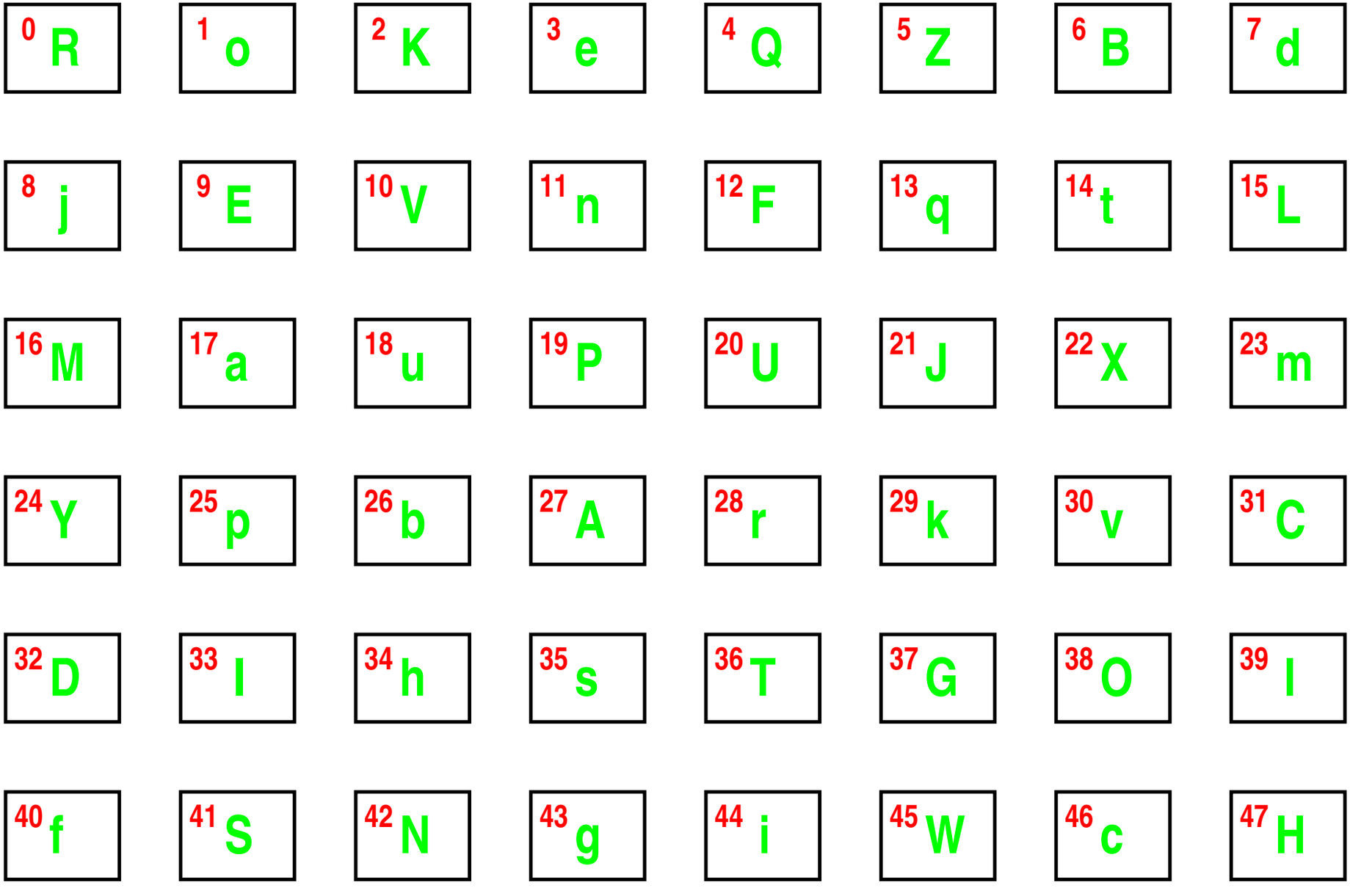
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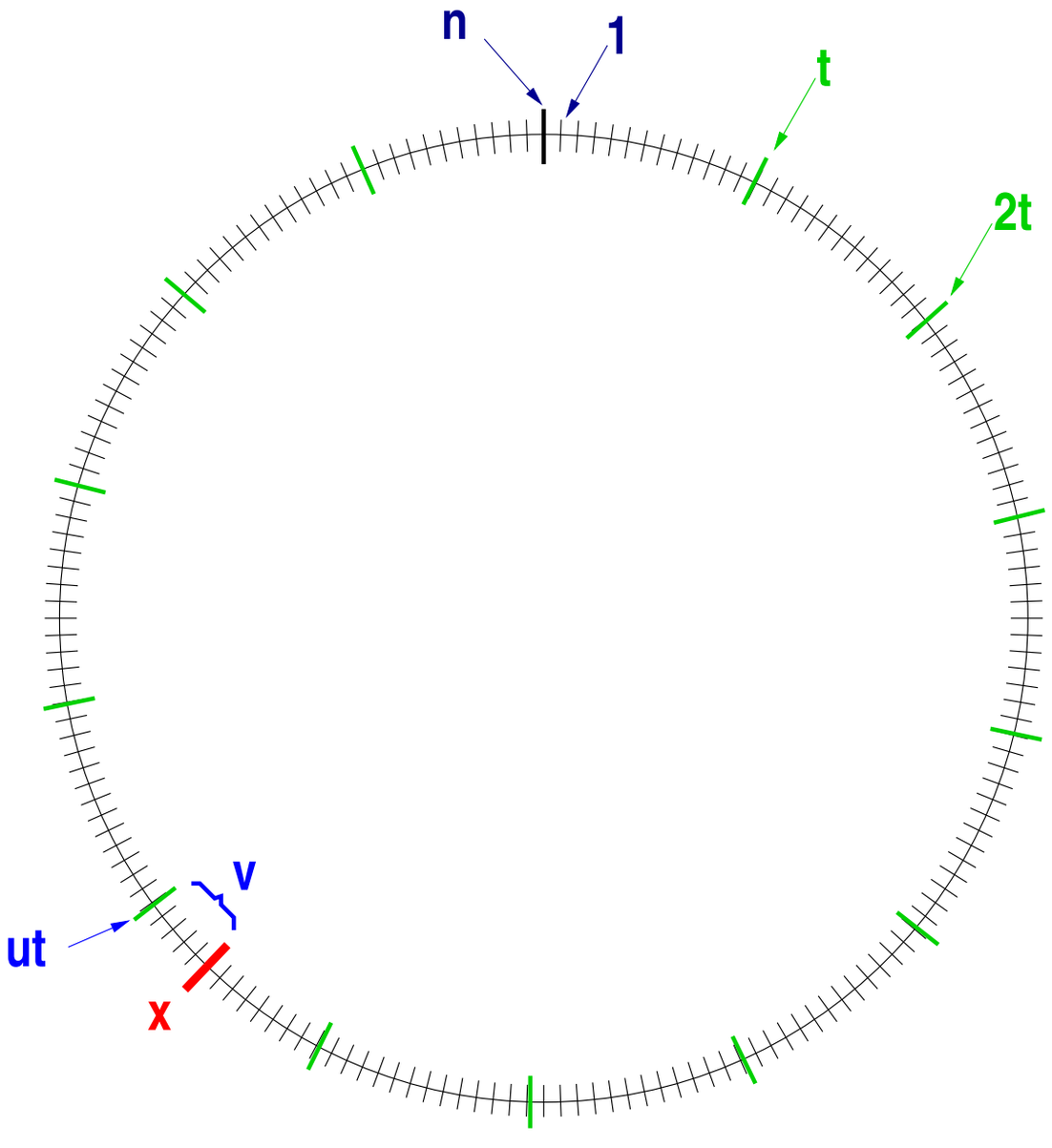
Baby-step giant-step DL algorithm



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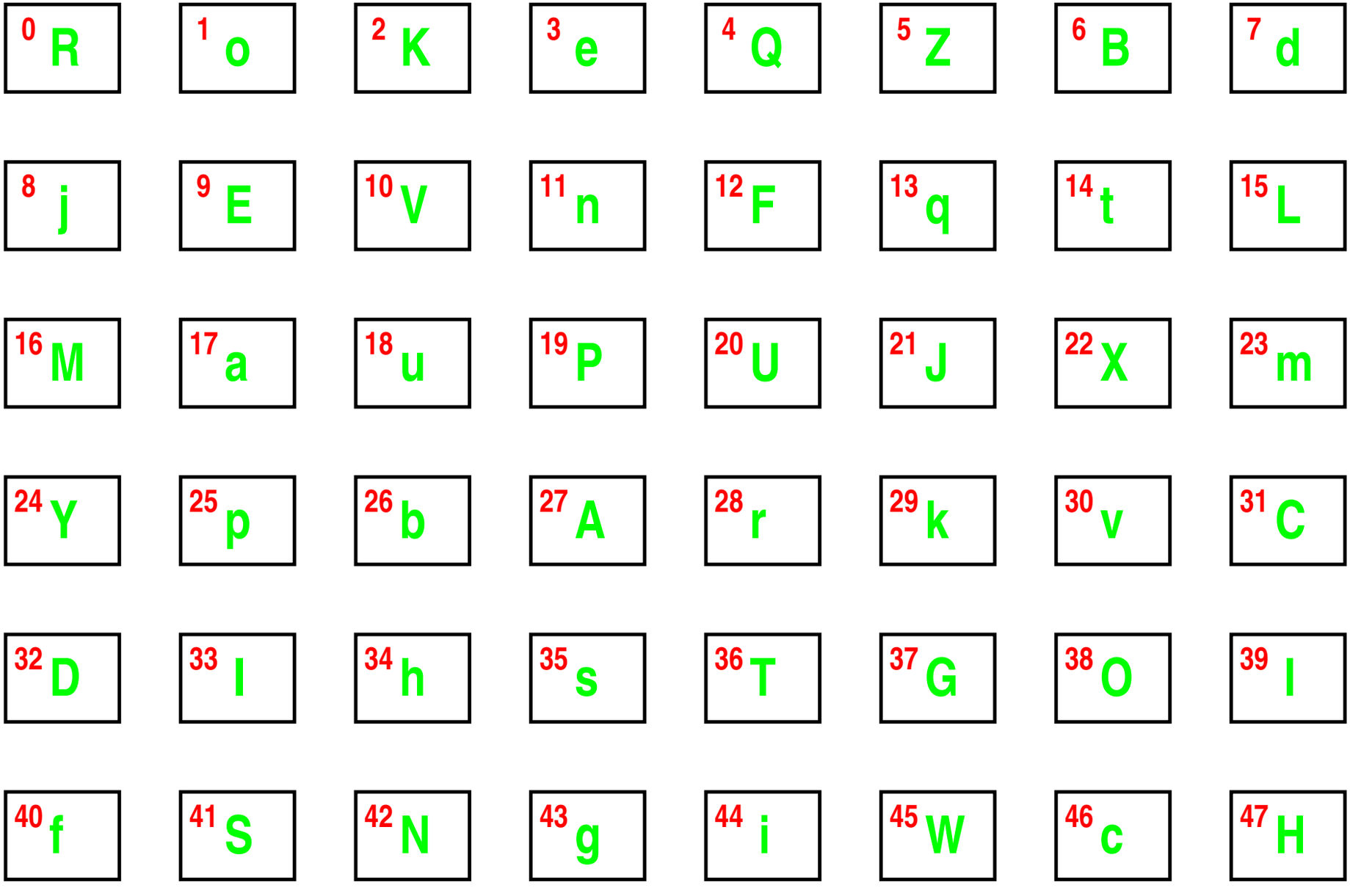
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Computational problems for a cyclic group

- **Discrete logarithm (DL):** $g^x \to x$
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Goals

- **Propose an abstract model of computation**
- **Capture reasonable restrictions on an algorithm's power**
- **Describe generic algorithms as special case, in a new and simpler model**
- **Derive lower bounds on computational problems and reductions**

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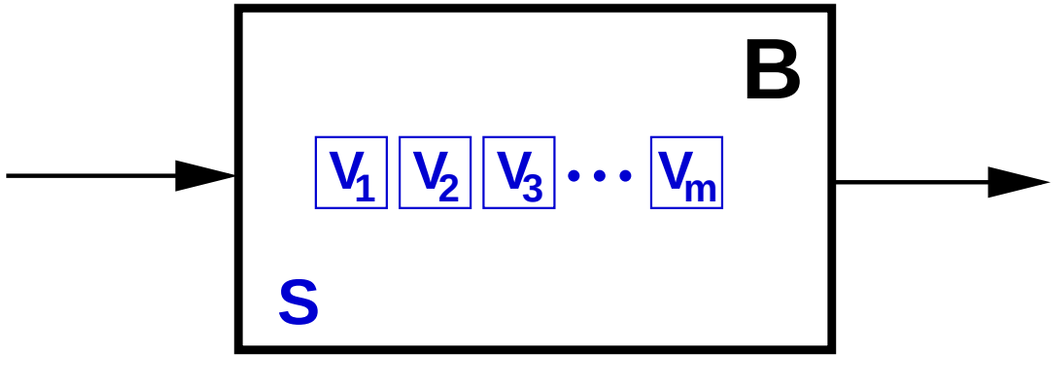
  Here: simpler model
- Goal: Prove lower bounds for DL etc.

Modeling generic algorithms



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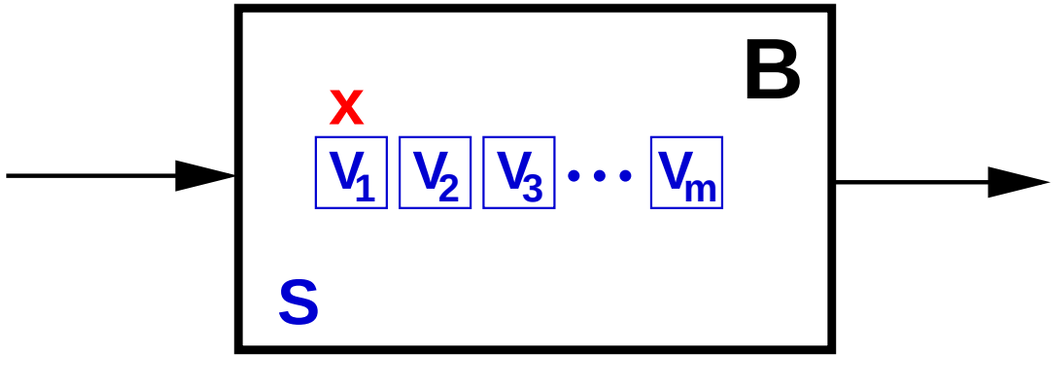

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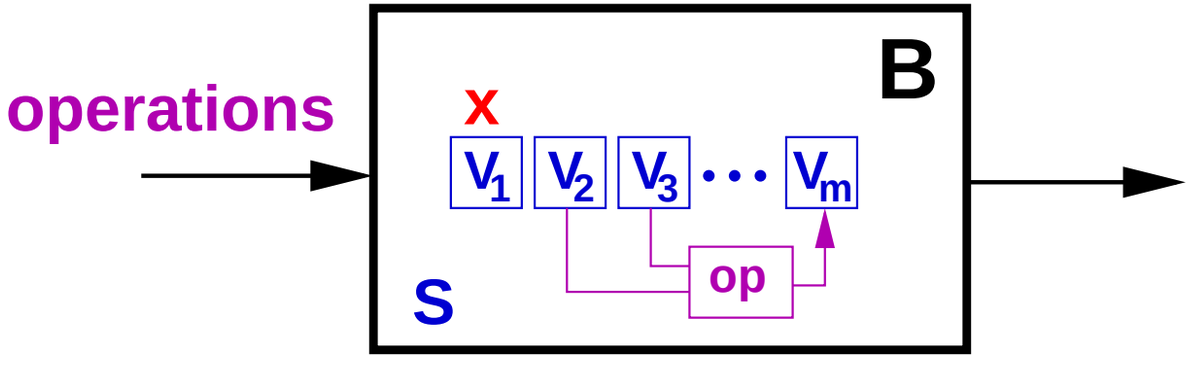

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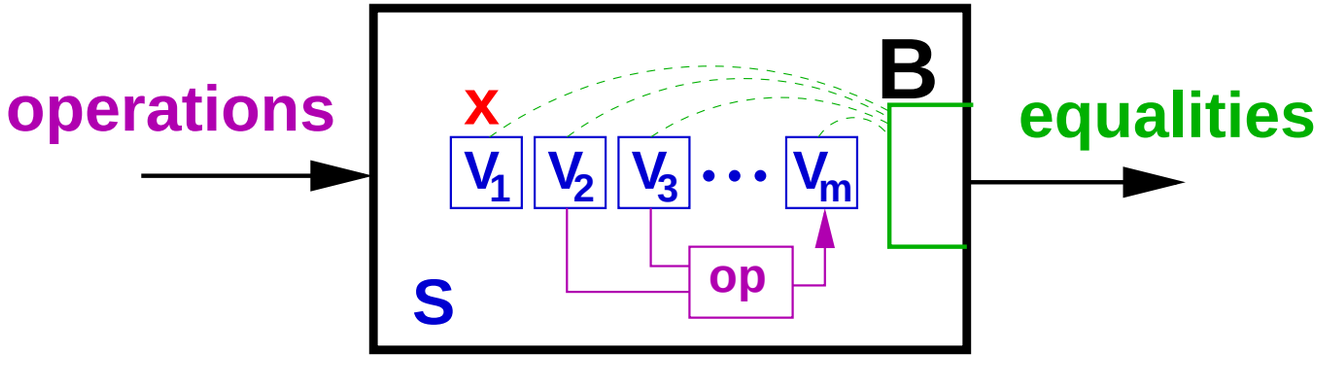

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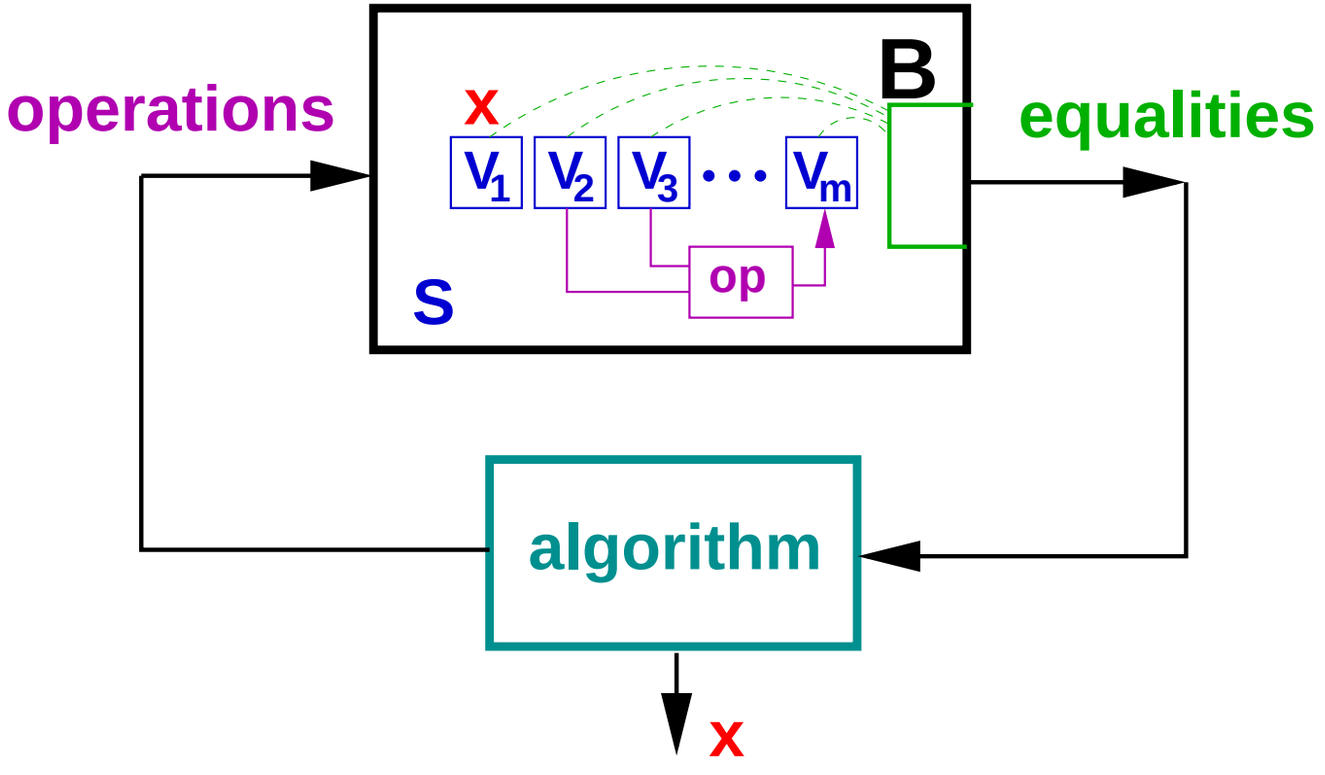

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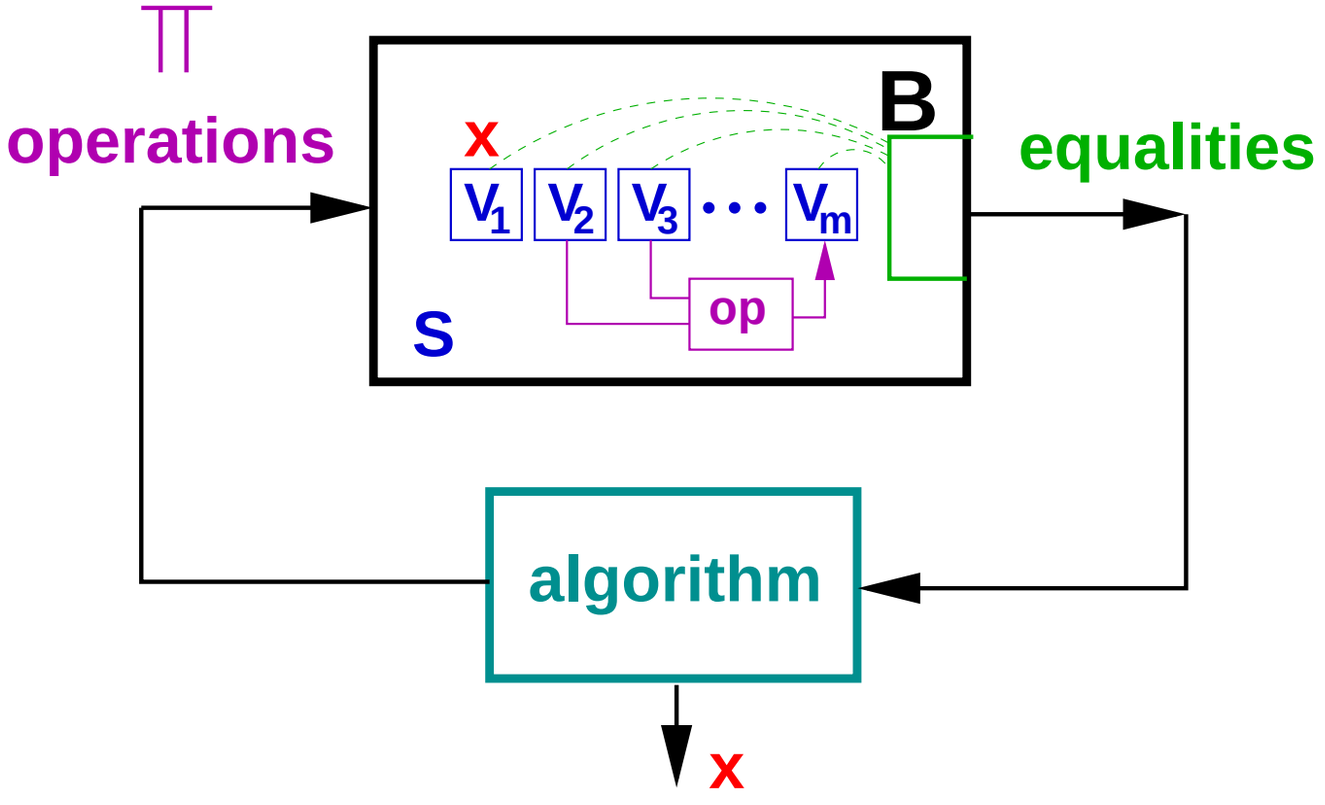

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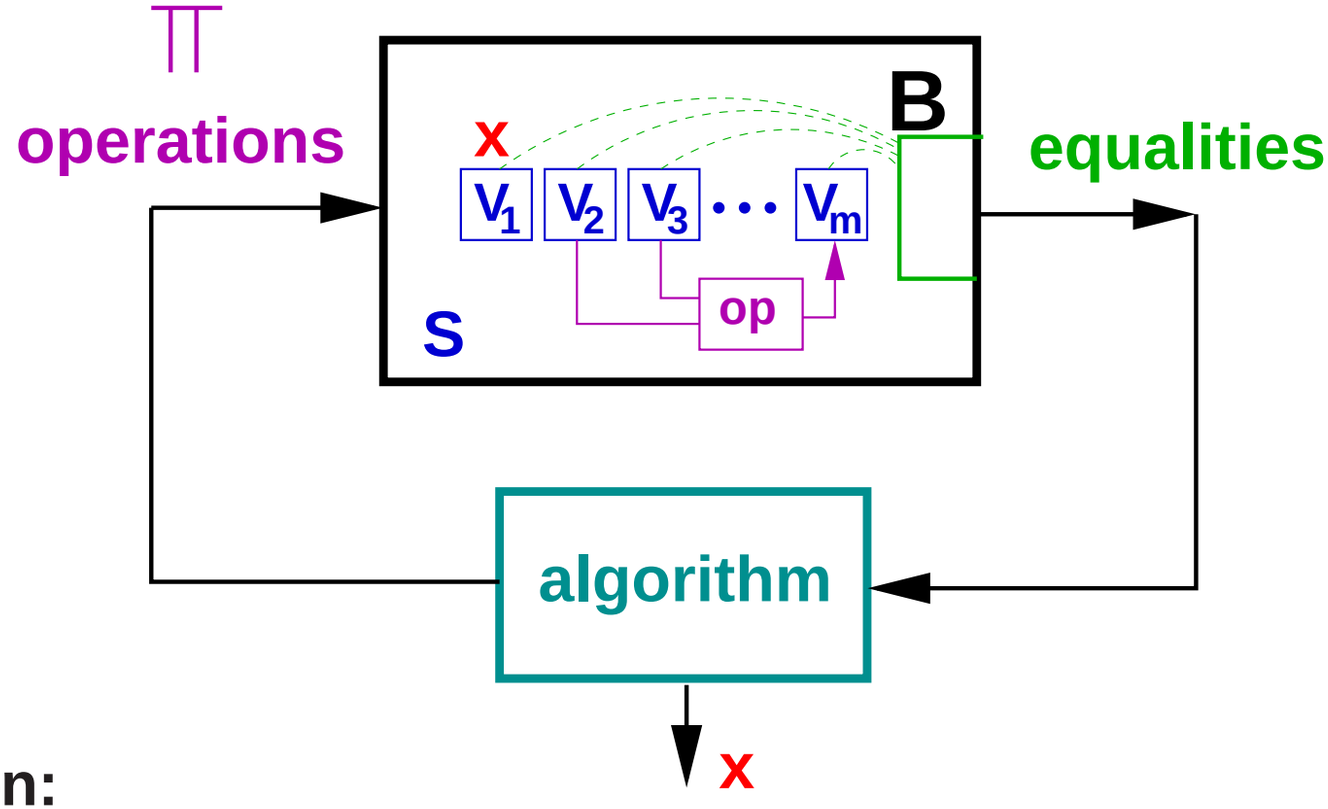
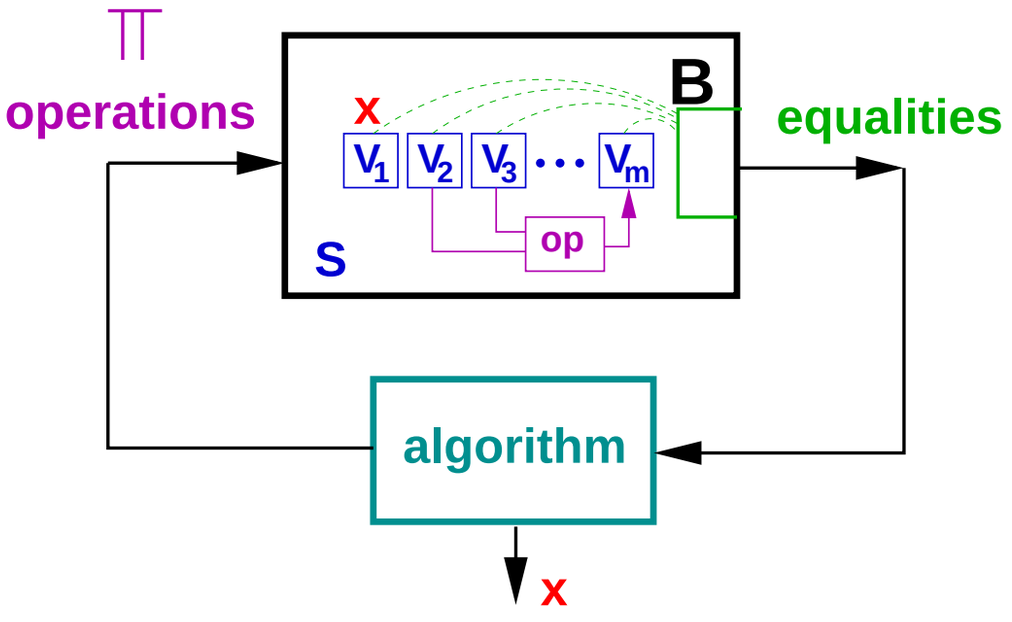


Notation:

- $|\mathbf{S}| = n$
- $\mathcal{C}$ = set of constant functions
- $k = \#$ queries
- $\mathbf{p_S}$ = success probability
- $\overline{\Pi}$ = set of functions computable by Π

Example: no structure assumed



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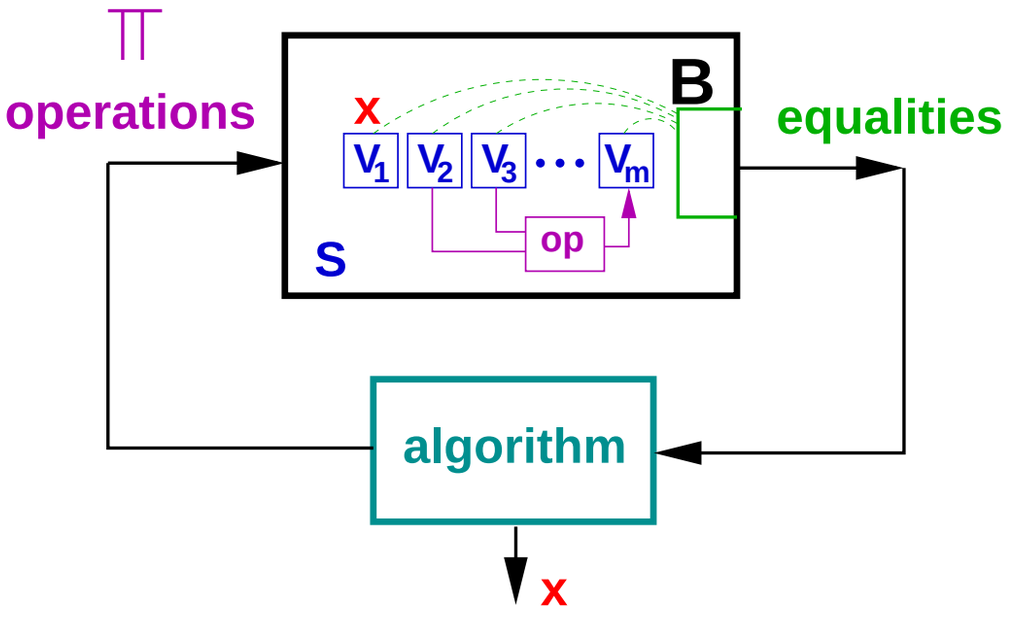

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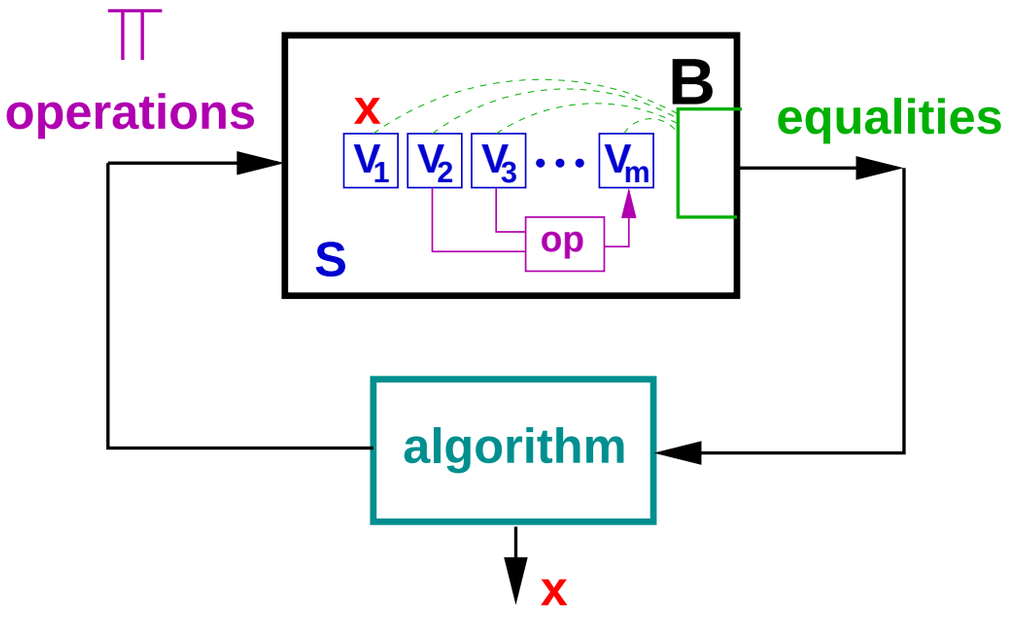

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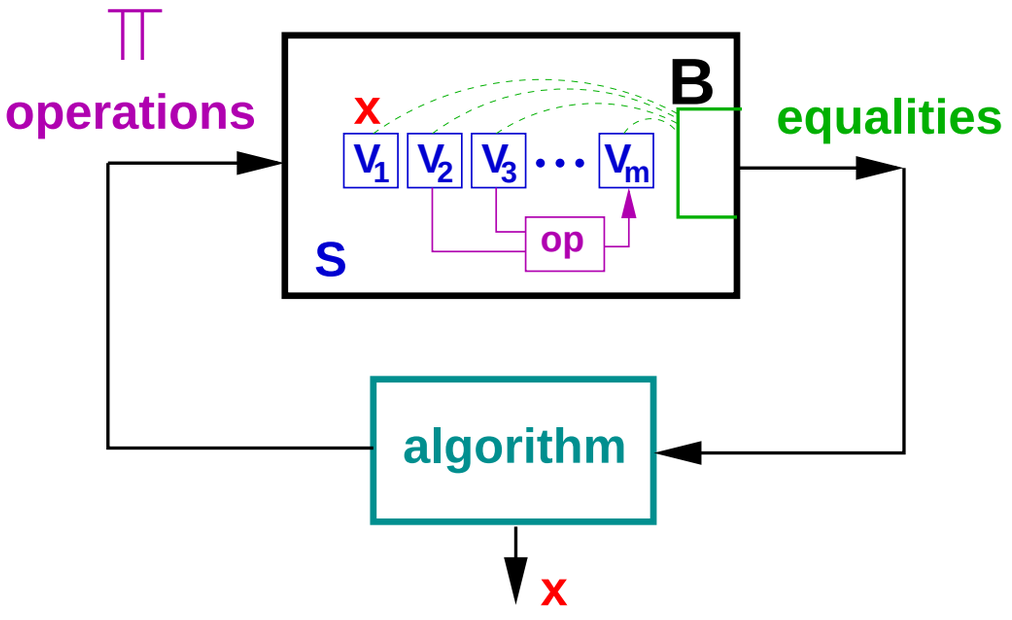
Proof: obvious?

Group action



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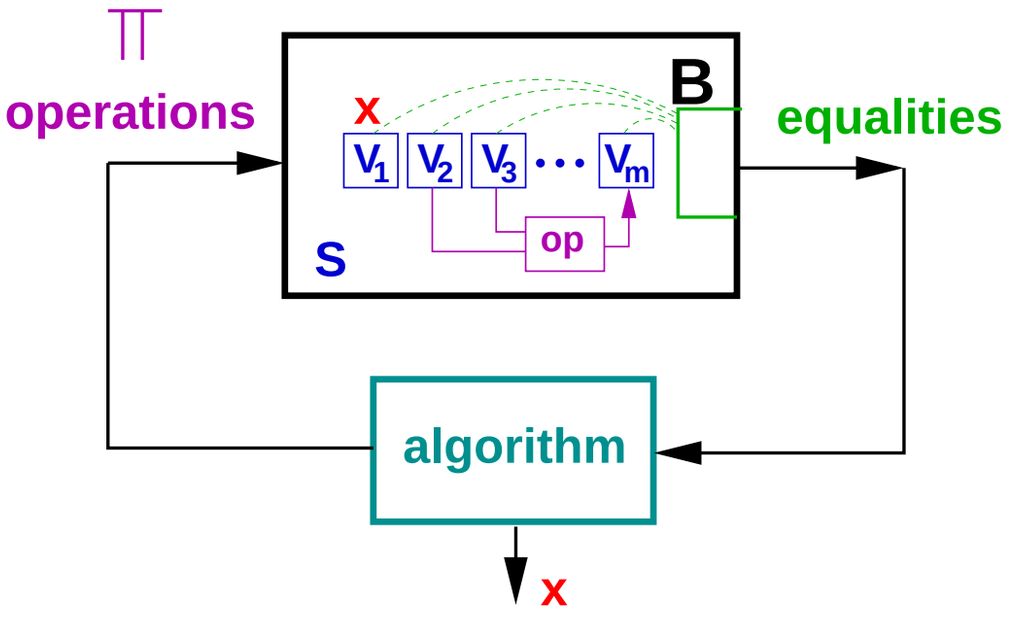

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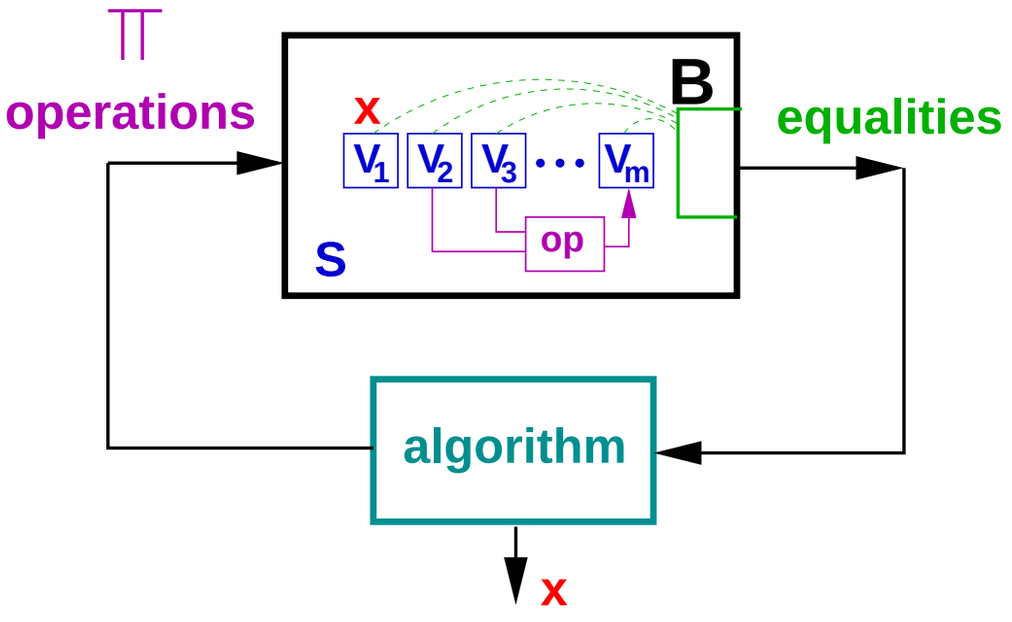

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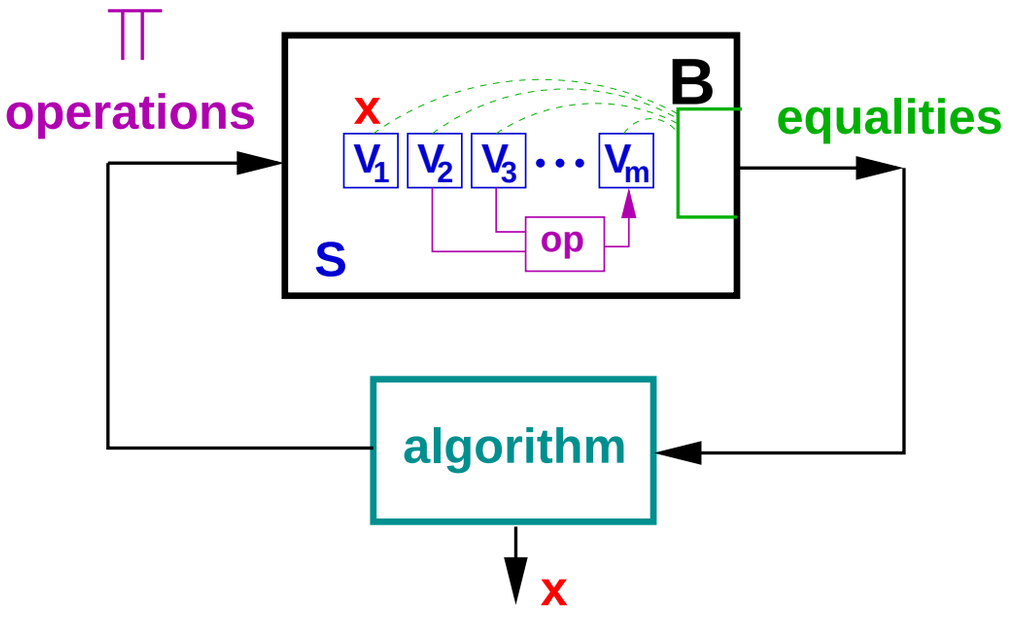

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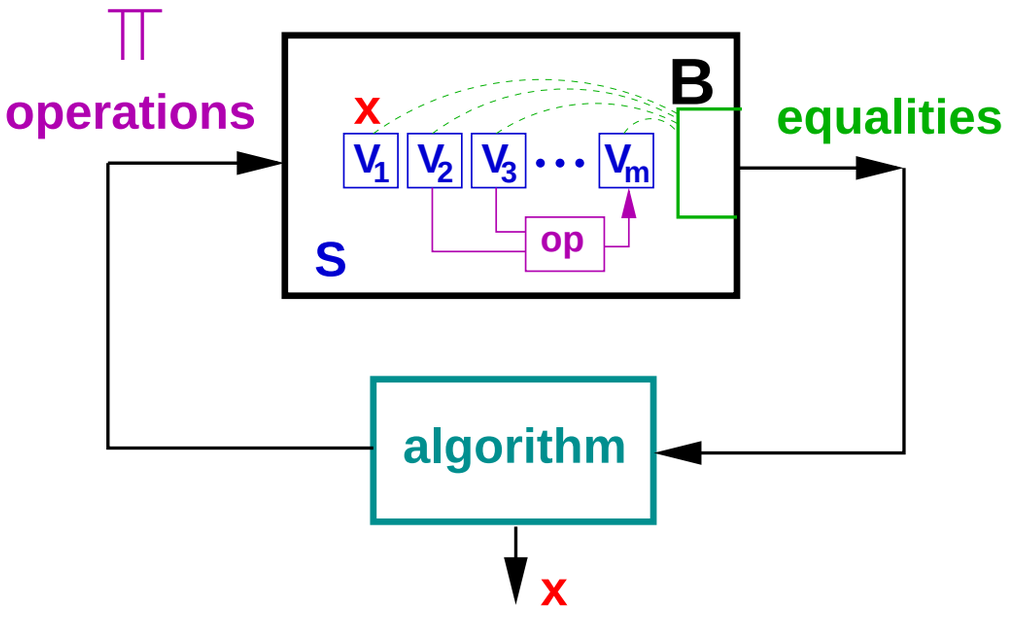

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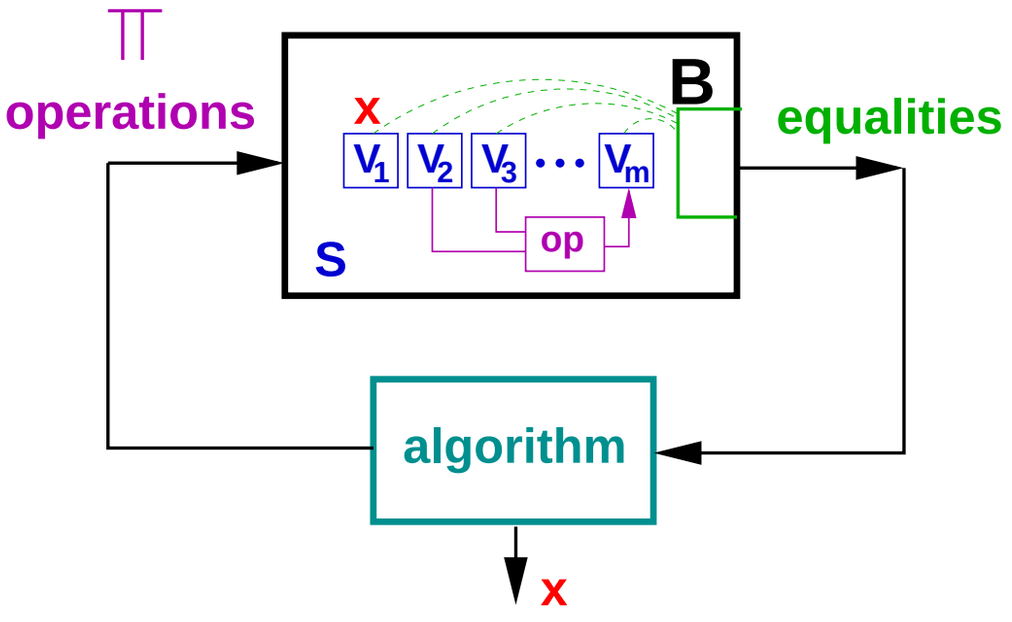

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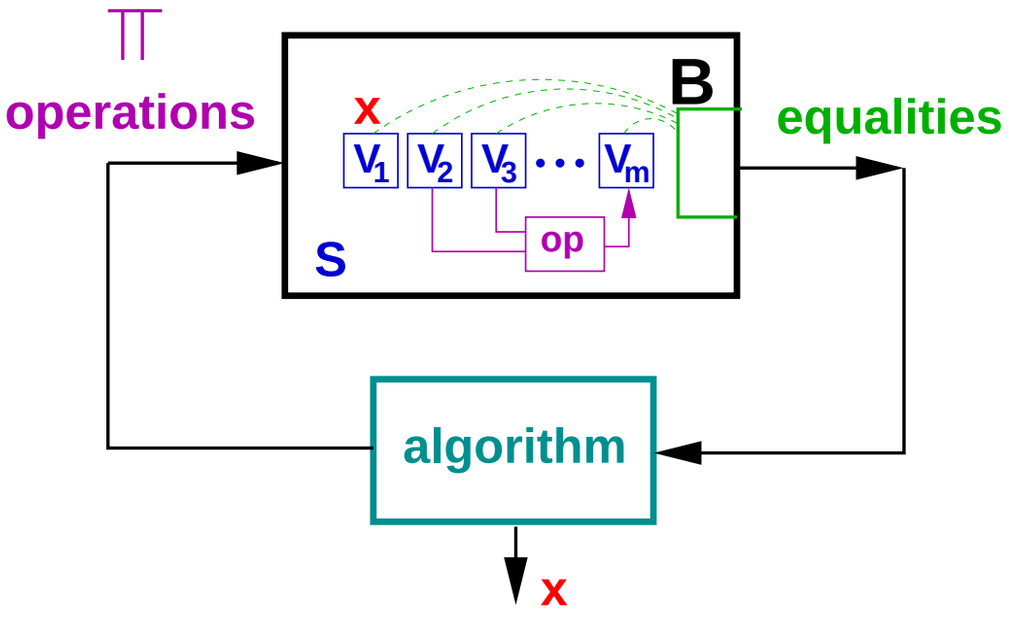

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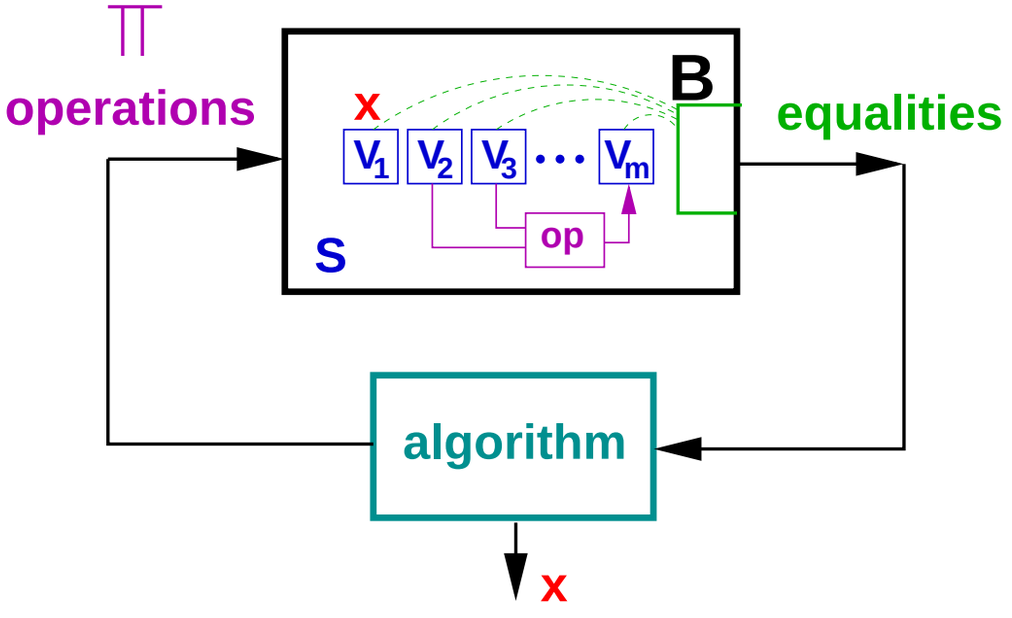

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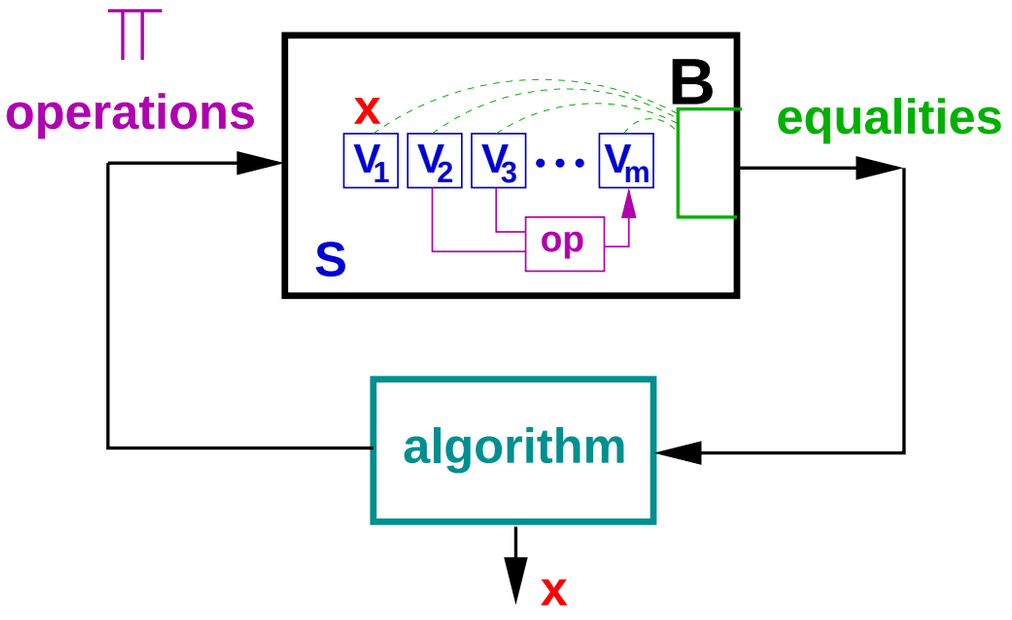
Group operation $\oplus$



$\mathbf{S} = \{0,1\}^\ell,\ n = 2^\ell$

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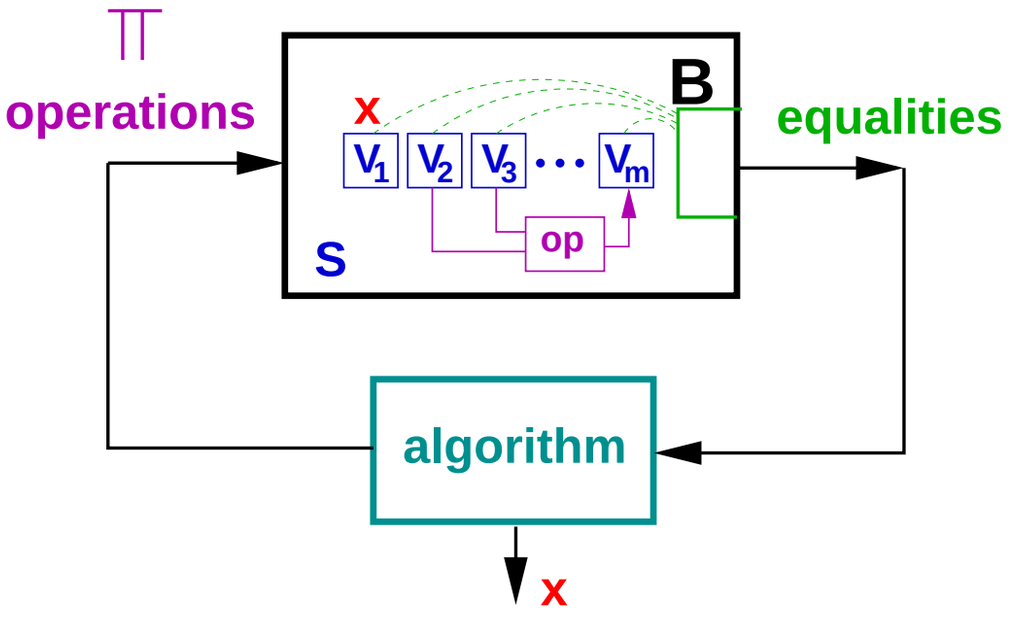

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Theorem: $\mathbf{p_S} \leq \frac{1}{4}k^2/n \quad \Rightarrow\ O(\sqrt{n})$ **lower bound**

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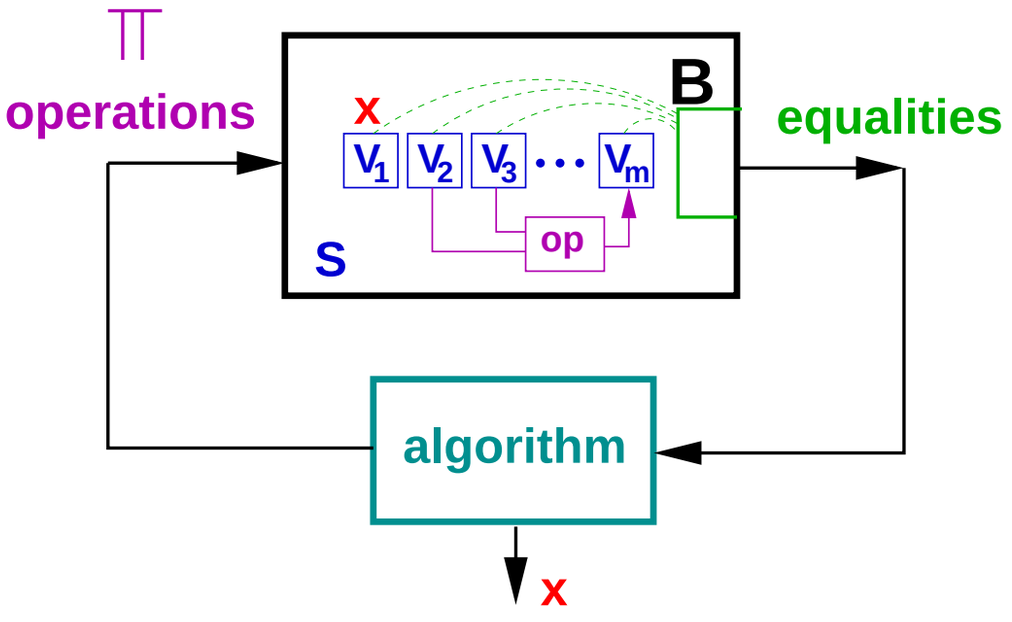

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Proof: $\overline{\Pi} = \{\mathbf{b} |\ \mathbf{b} \in \mathbf{S}\} \cup \{\mathbf{x} \oplus \mathbf{a} \mid \mathbf{a} \in \mathbf{S}\}$

Same argument as before.

Polynomial equations modulo q

Lemma: **The fraction of solutions $(x_1, \ldots, x_k) \in \mathbf{Z}_q$ of the multivariate polynomial equation**

$$p(x_1, \ldots, x_k) \equiv_q 0$$

of degree d is at most d/q.

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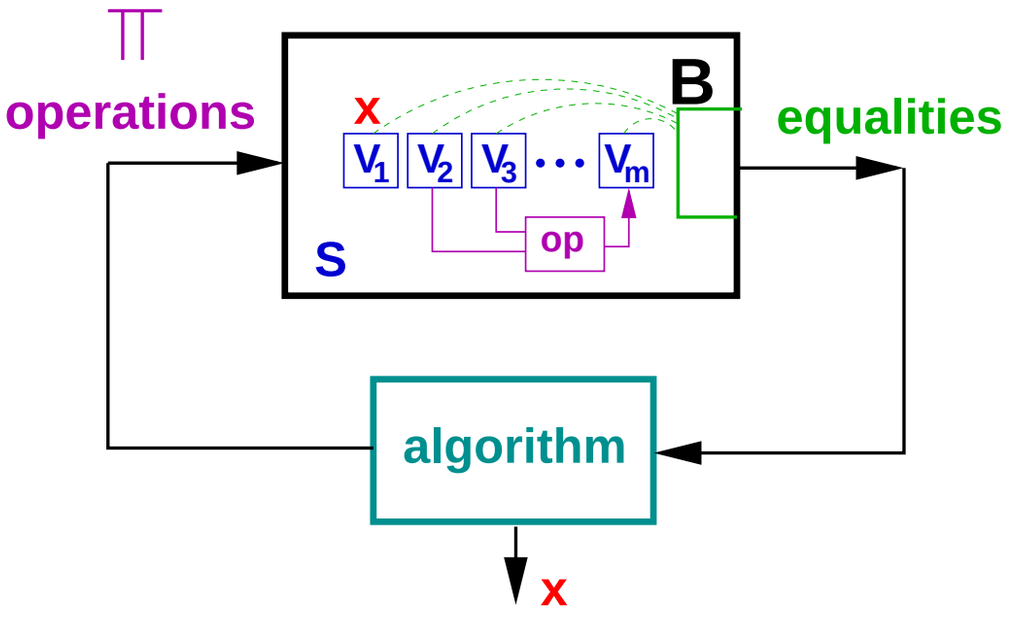
Proof sketch:

- **q prime $\Rightarrow$ $\mathbb{Z}_q$ is a field.**
- **univariate degree-d polynomial $p(x)$: # of roots $\leq d$ (unless $p(x)$ is the 0-polynomial)**
- **Generalize to multivariate polynomials**

Group operation $+$



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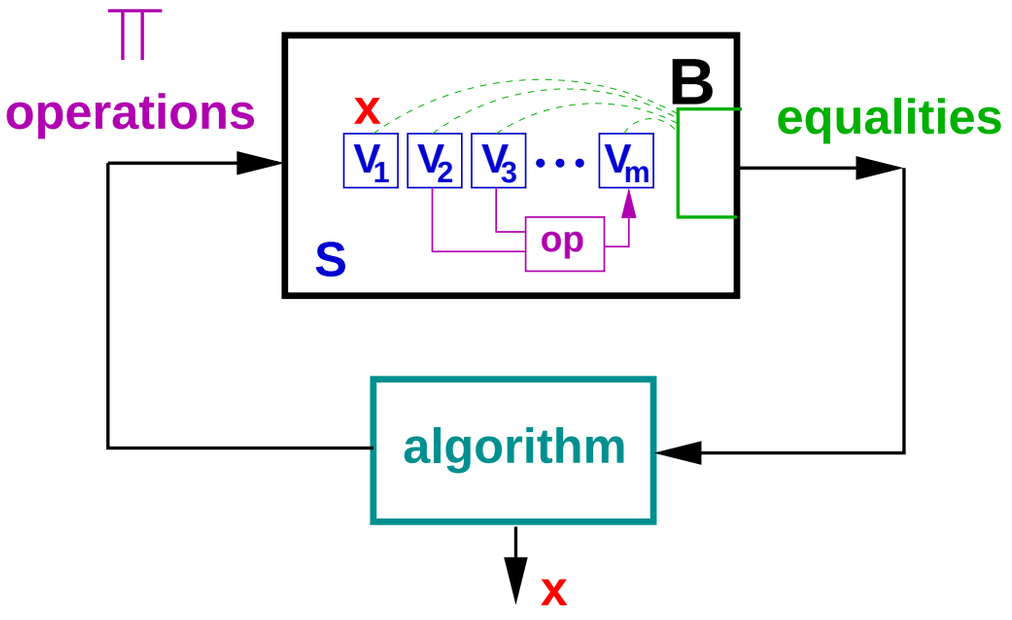

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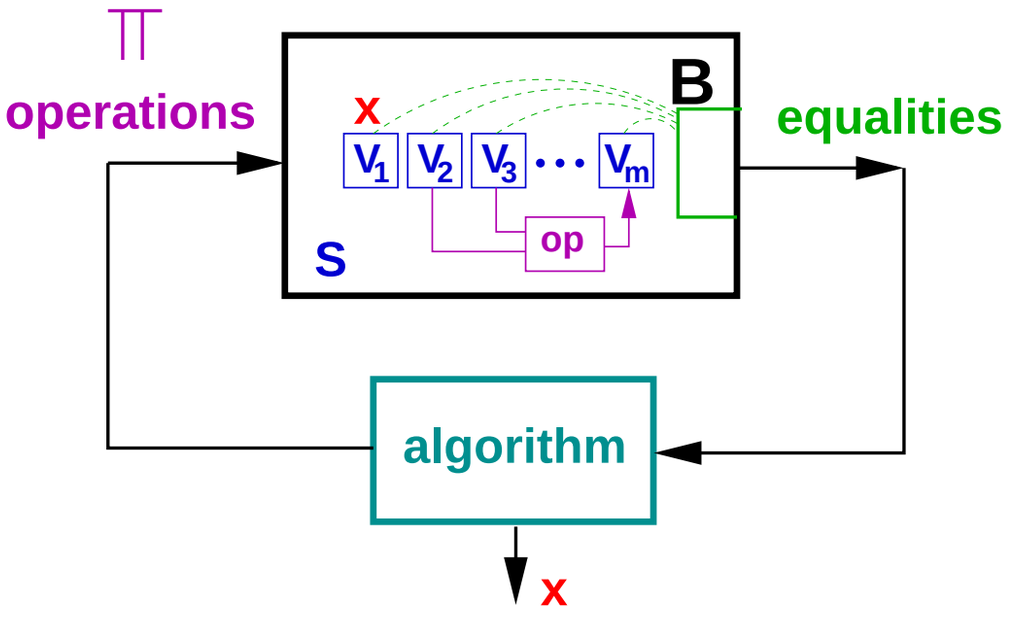
Group operation $+$: Generic l. b. for DL



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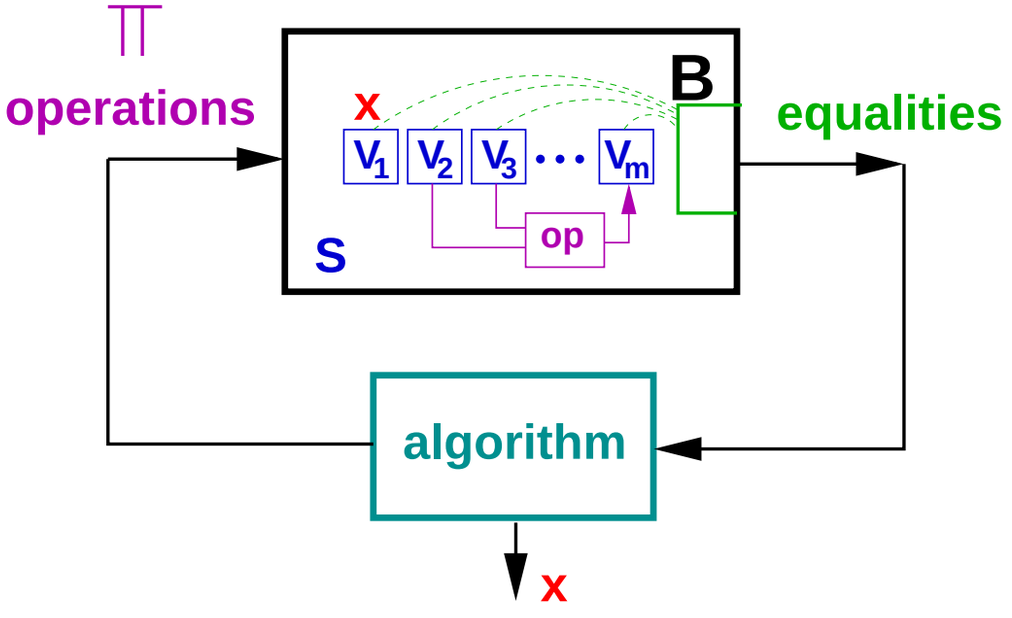

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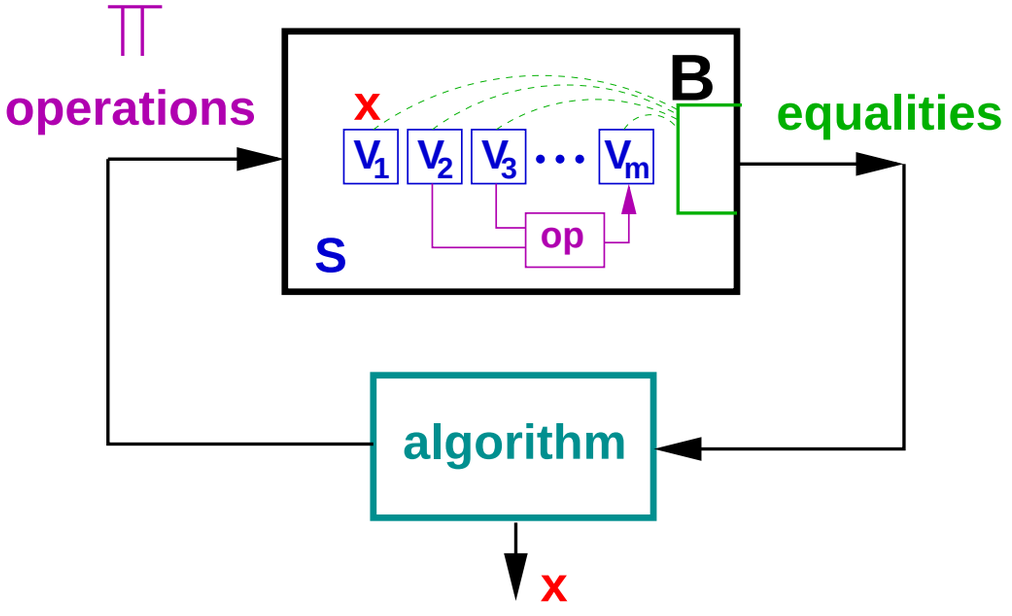

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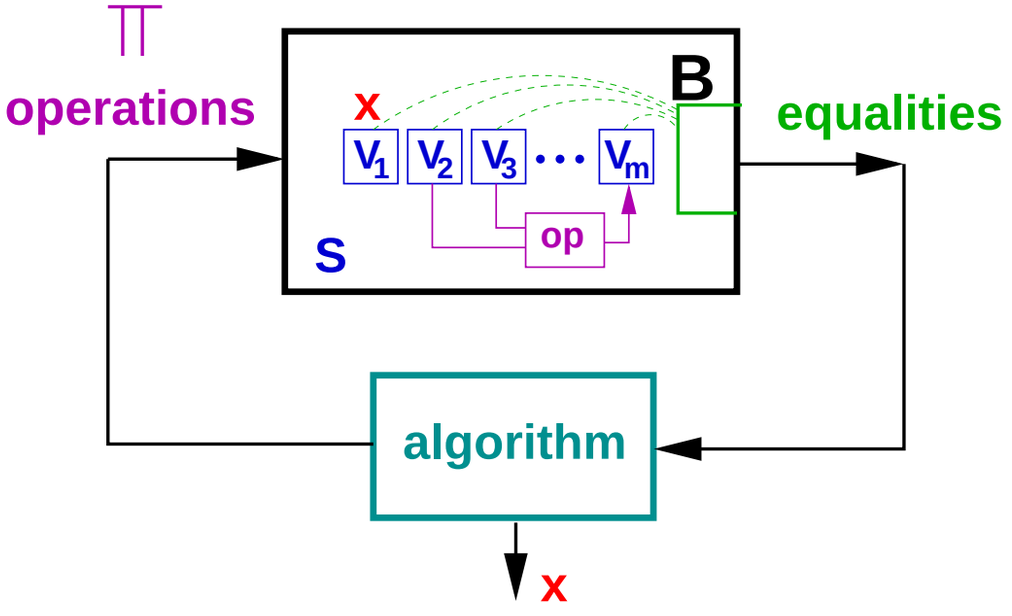

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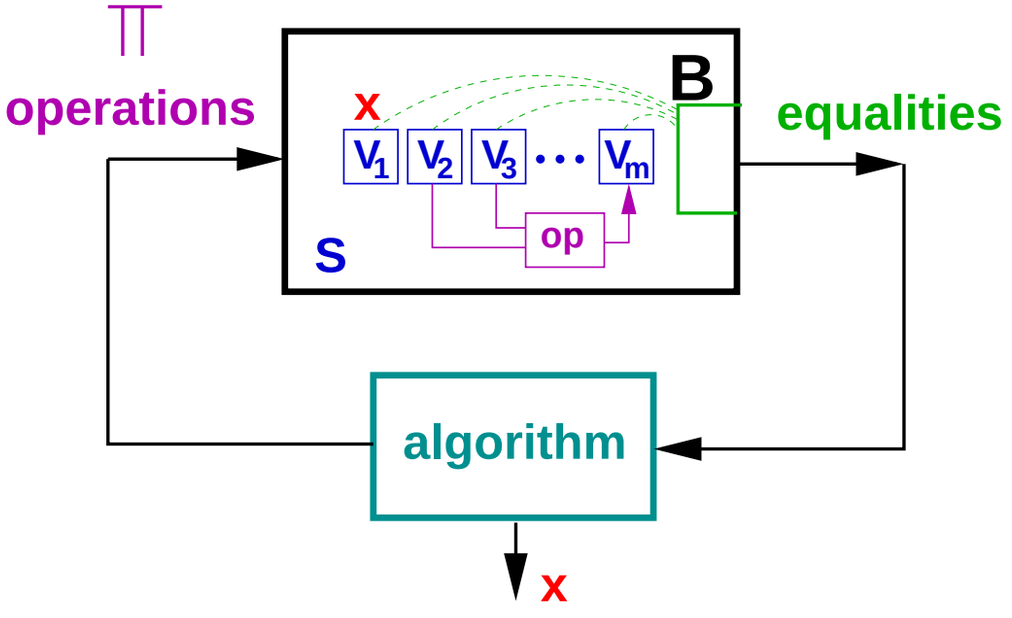

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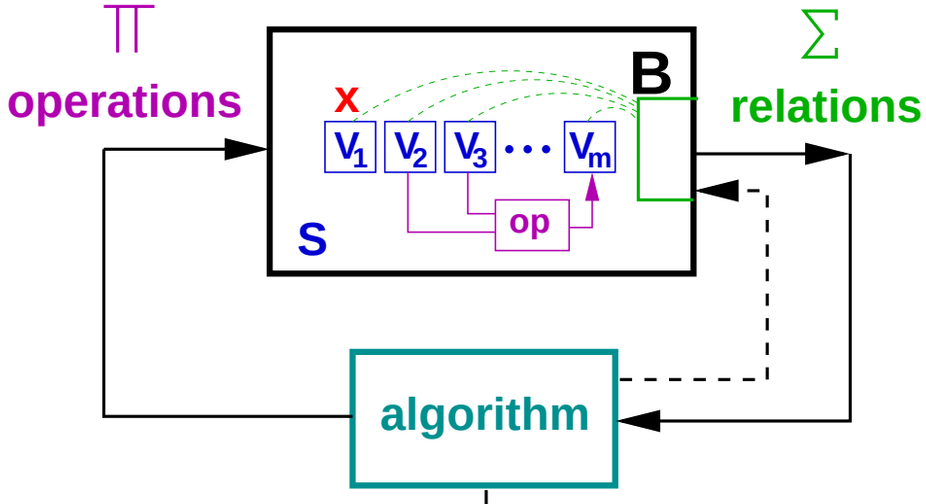
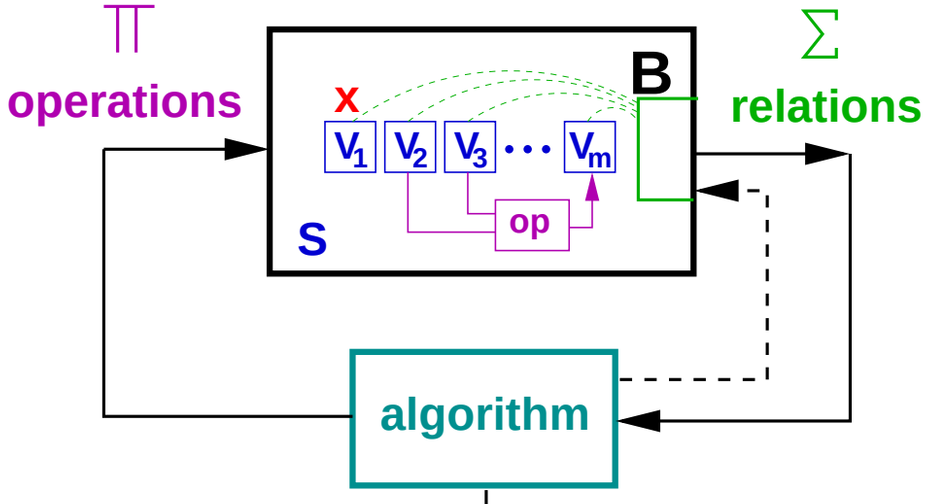
# of solutions **x** bounded by # of pairs (i, j): $\binom{k}{2}$

Three types of problems



Initial state: $V_1, \ldots, V_d$

Three types of problems



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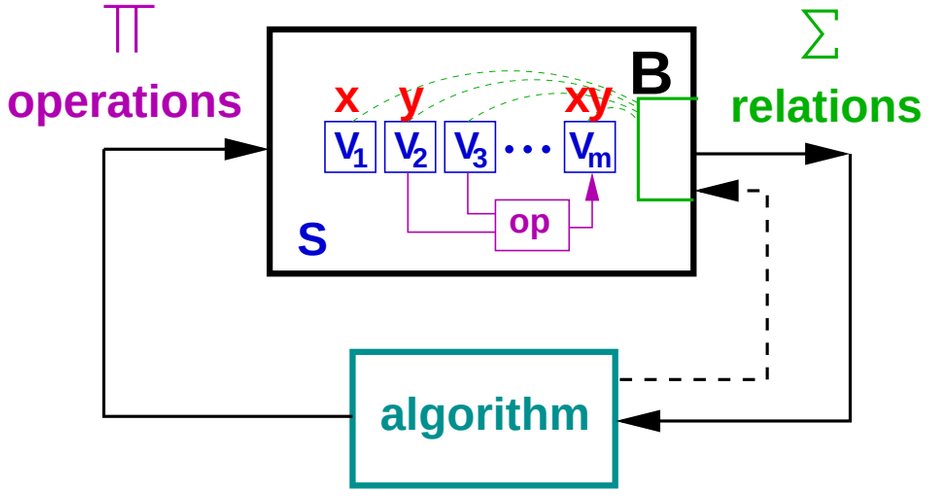

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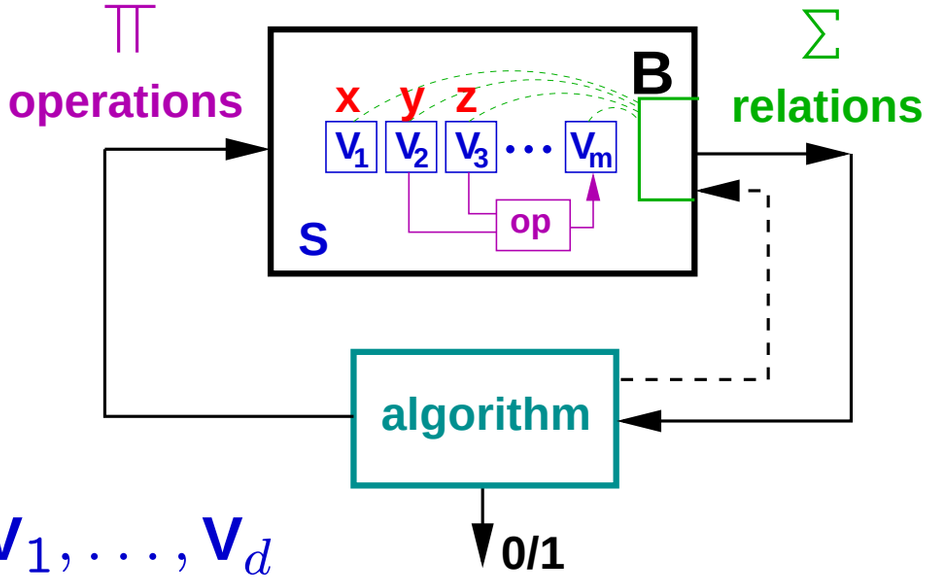

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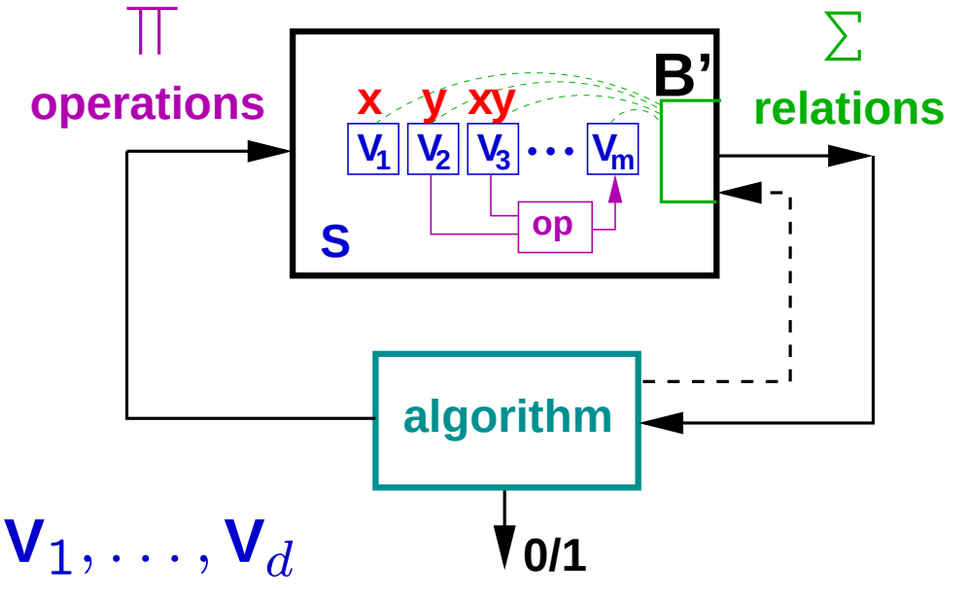
- **Distinction:** Distinguish two black-boxes B and B' of the same type with different distributions of the initial state. Example DDH: $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ vs. $(\mathbf{x}, \mathbf{y}, \mathbf{xy})$

Three types of problems



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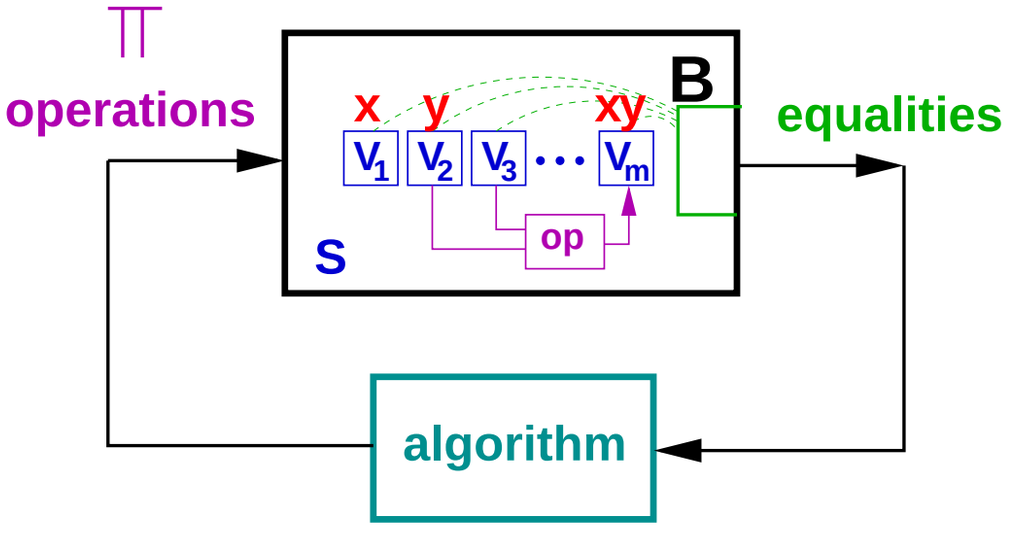
Generic lower bound for CDH

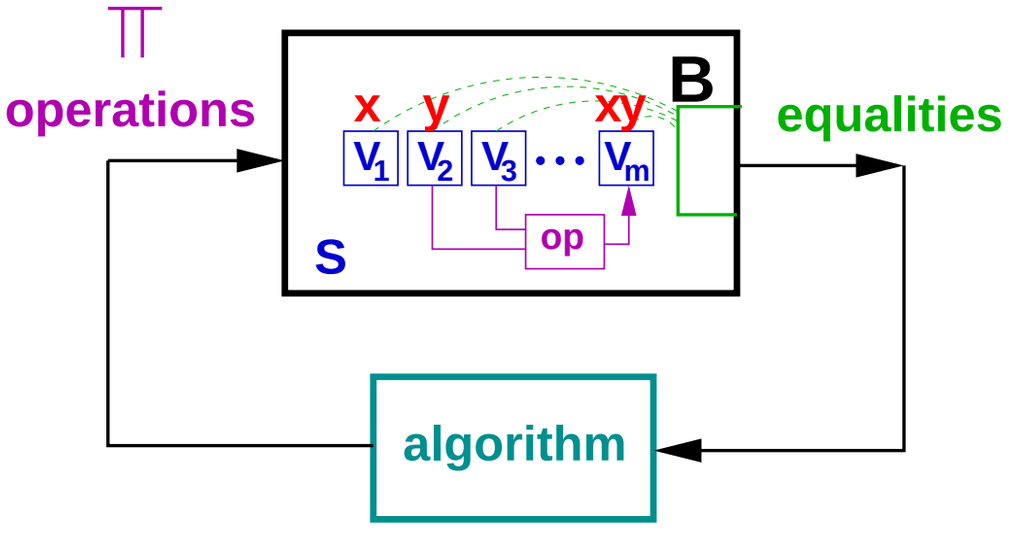


S $= Z_n$**, q larg. p.f. of n**

$\Pi = \{1, +\}$

Goal: **x**, **y** $\rightarrow$ **xy**

Generic lower bound for CDH



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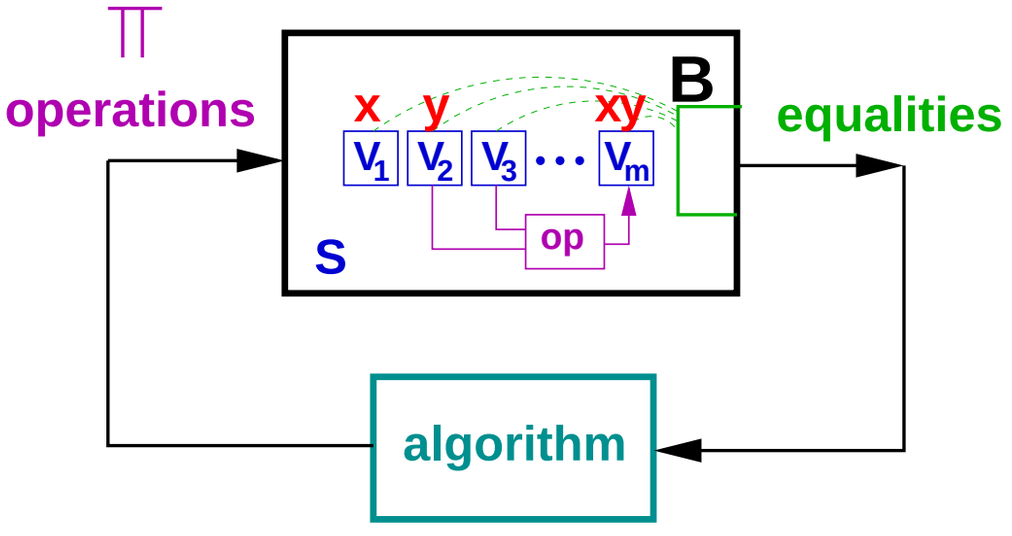

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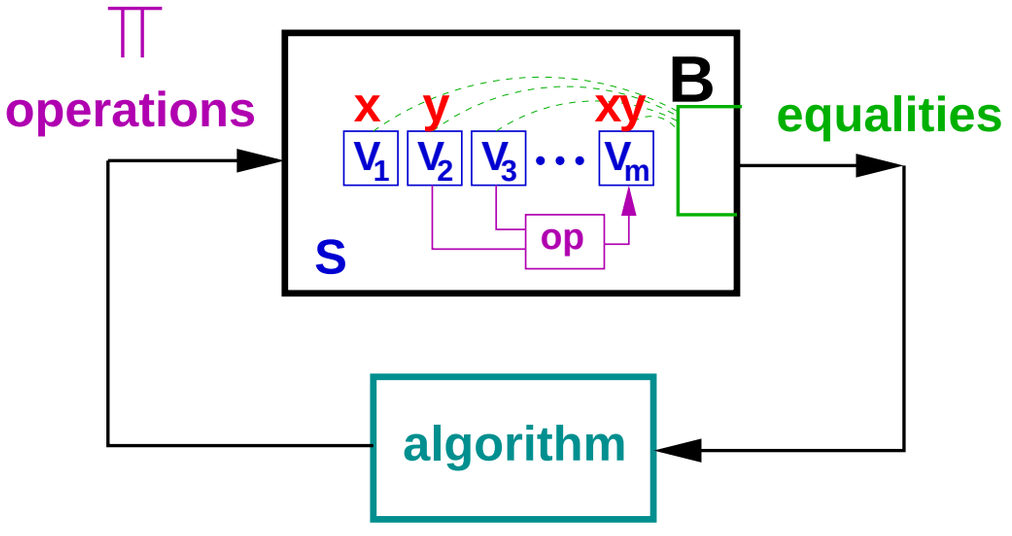

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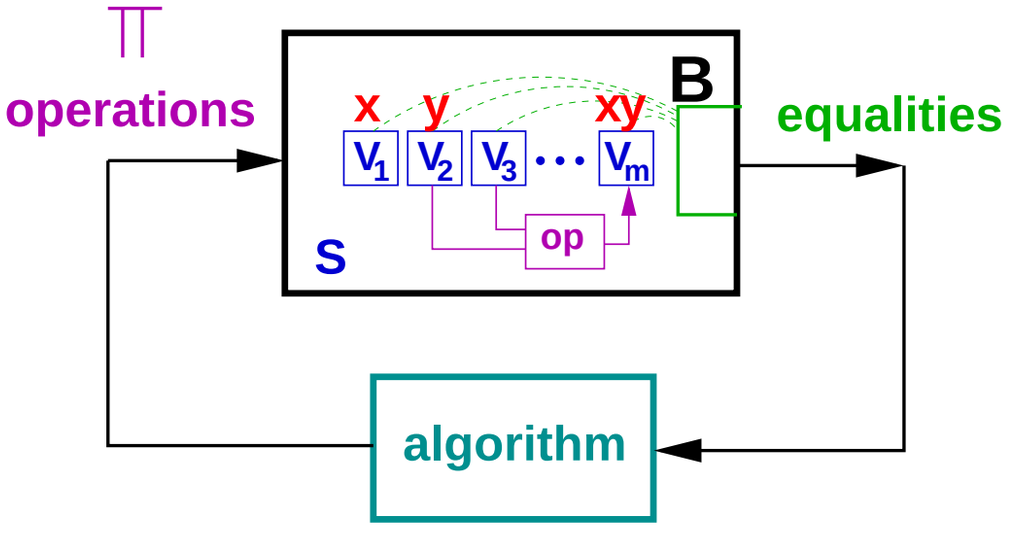

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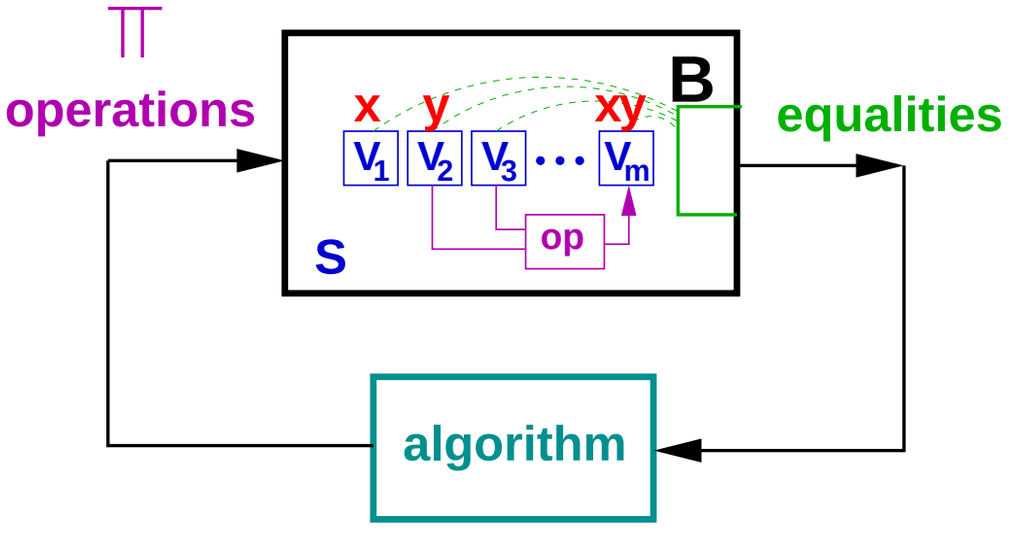

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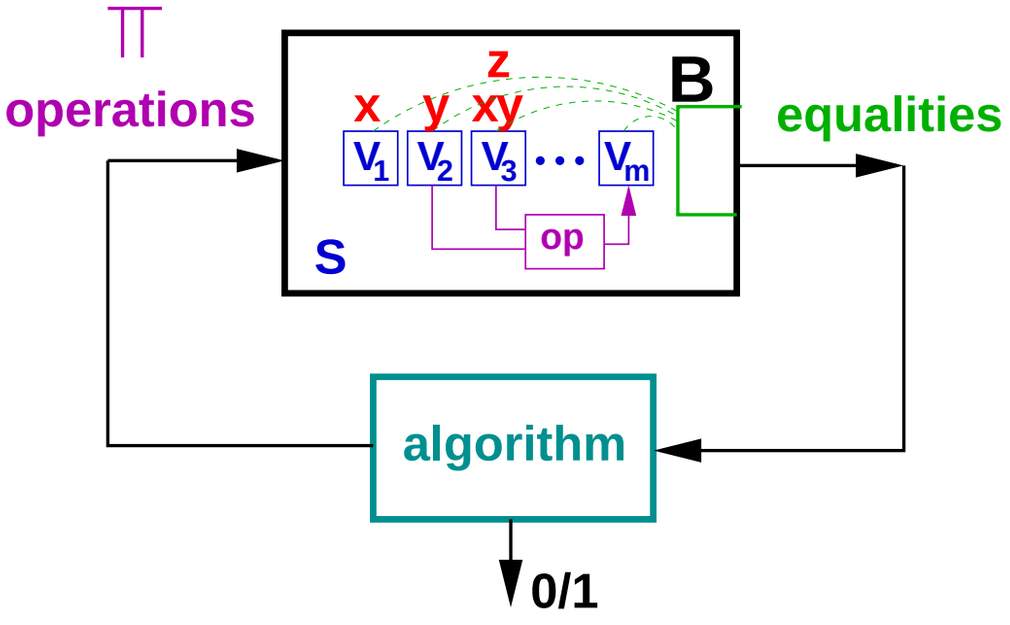
Fraction of pairs $(\mathbf{x}, \mathbf{y})$ **giving a collision** $\leq \left(\binom{k}{2} + 2k\right)/q$

Generic lower bound for DDH



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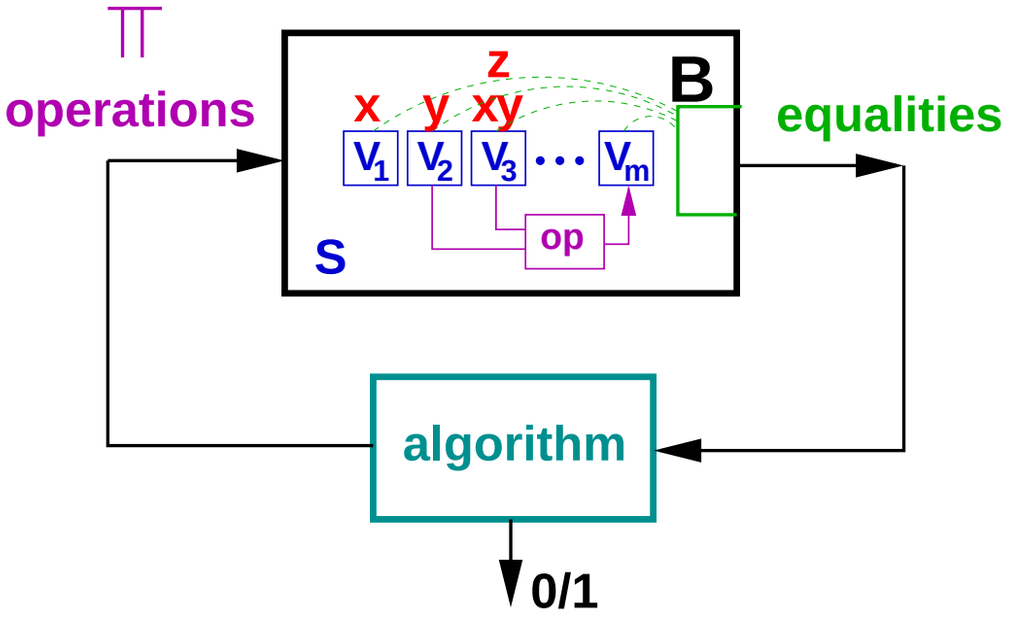

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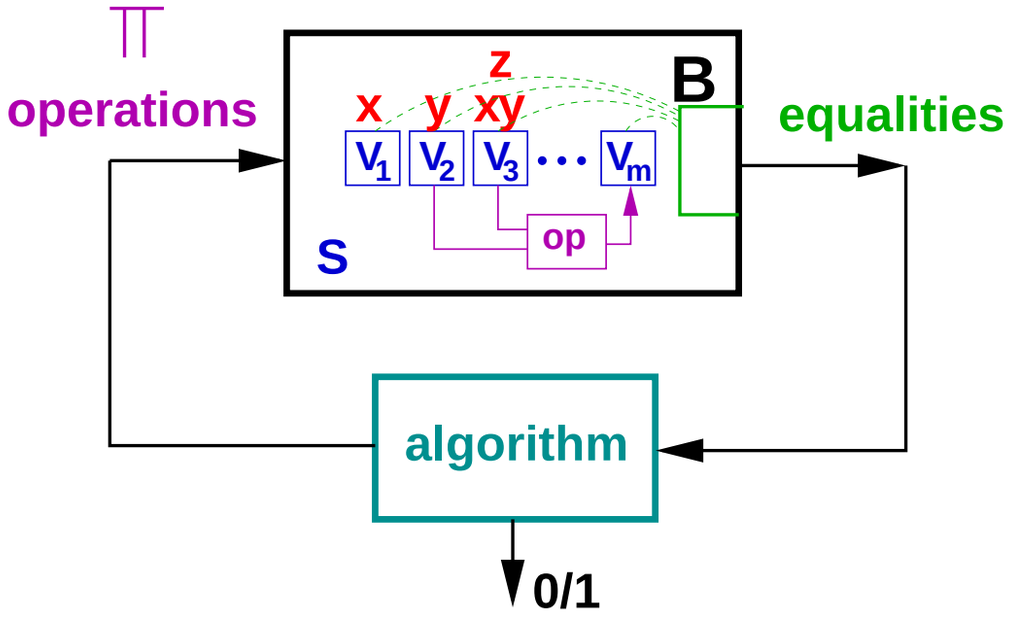

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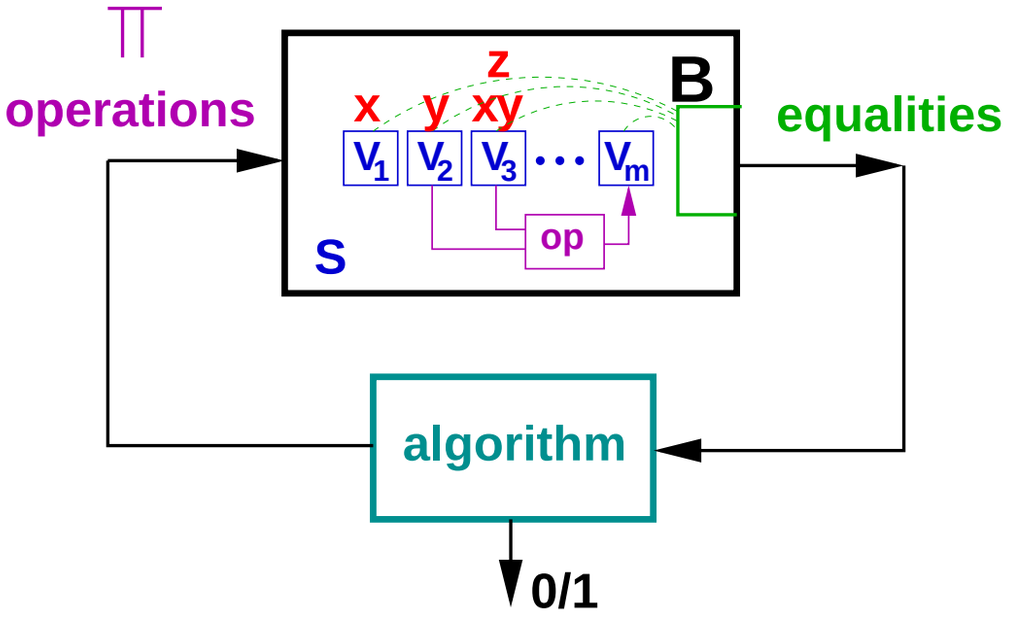

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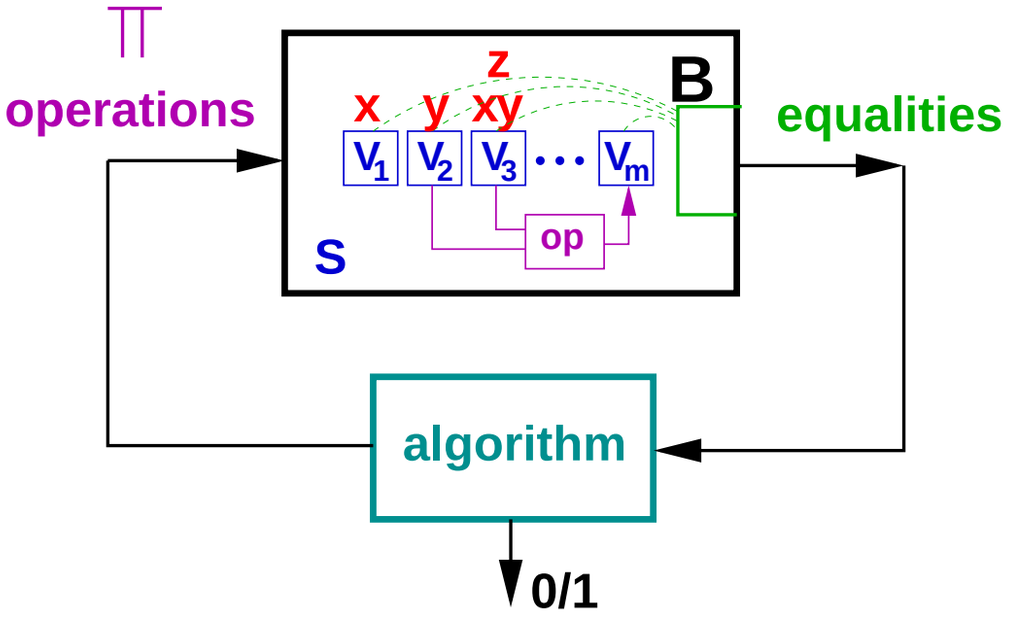

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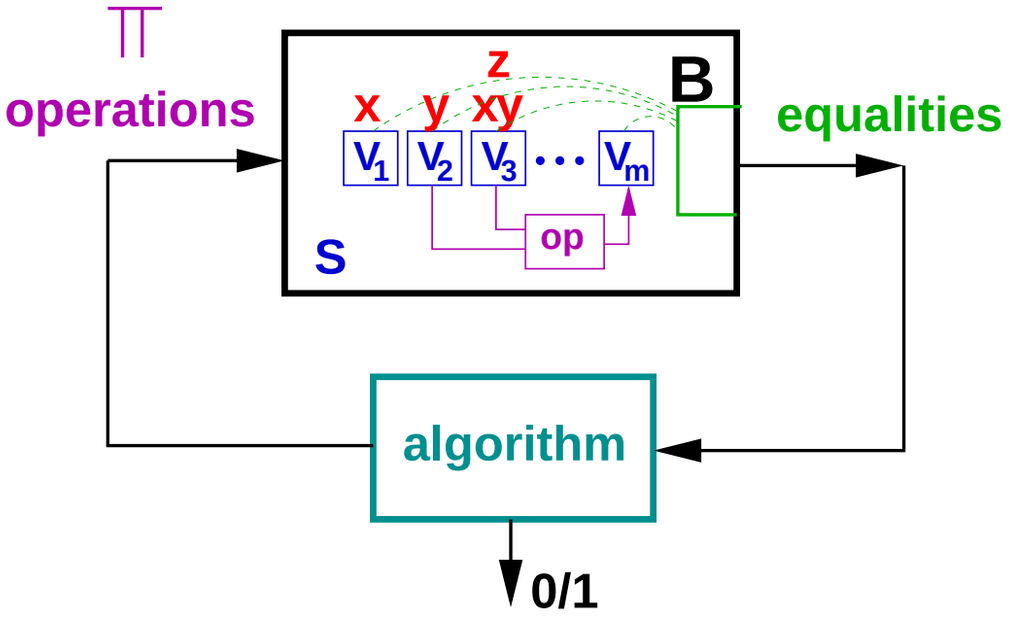
Generic lower bound for DDH



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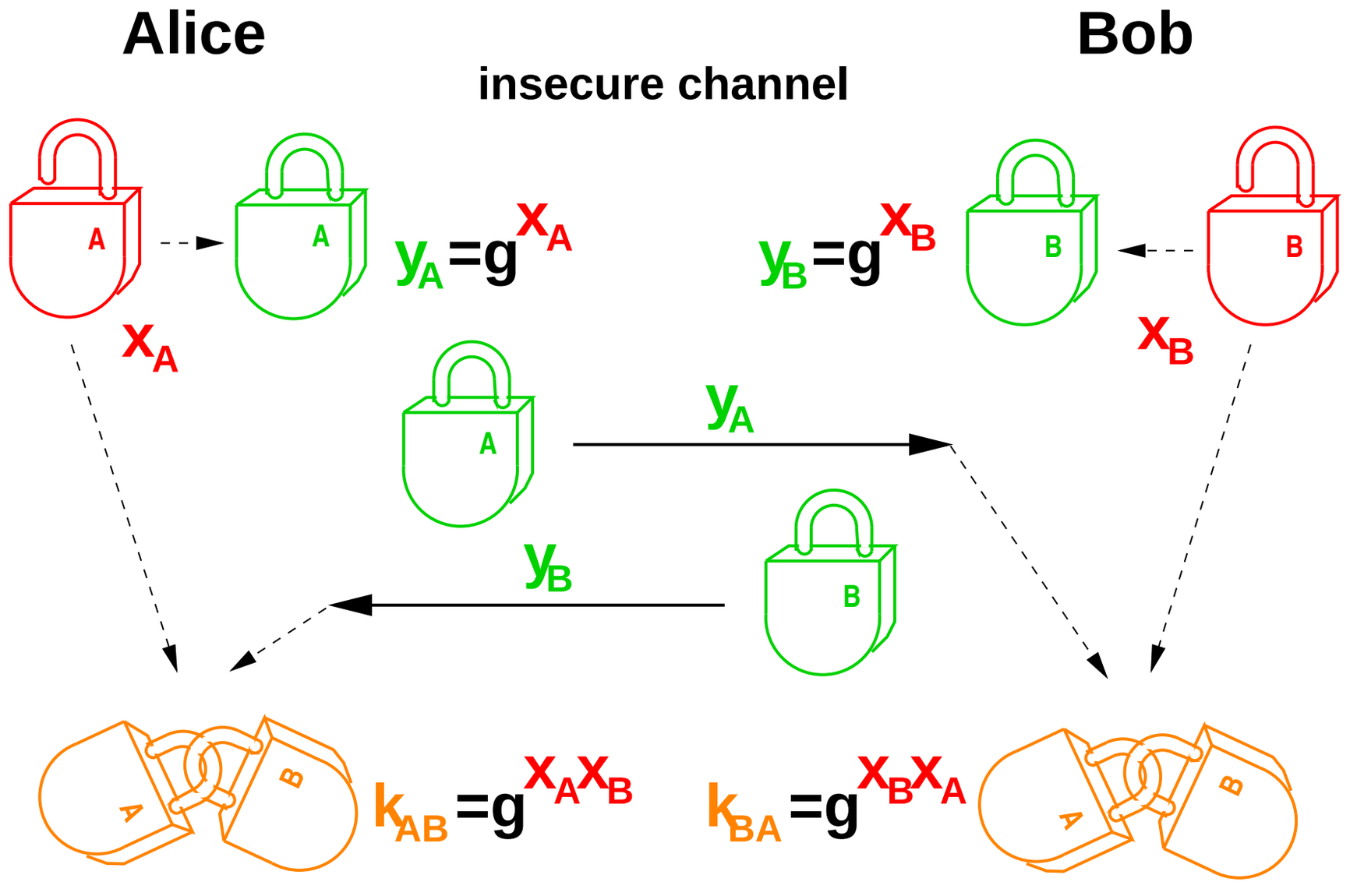
The Equivalence of Breaking Diffie-Hellman and Computing Discrete Logarithms

Ueli Maurer

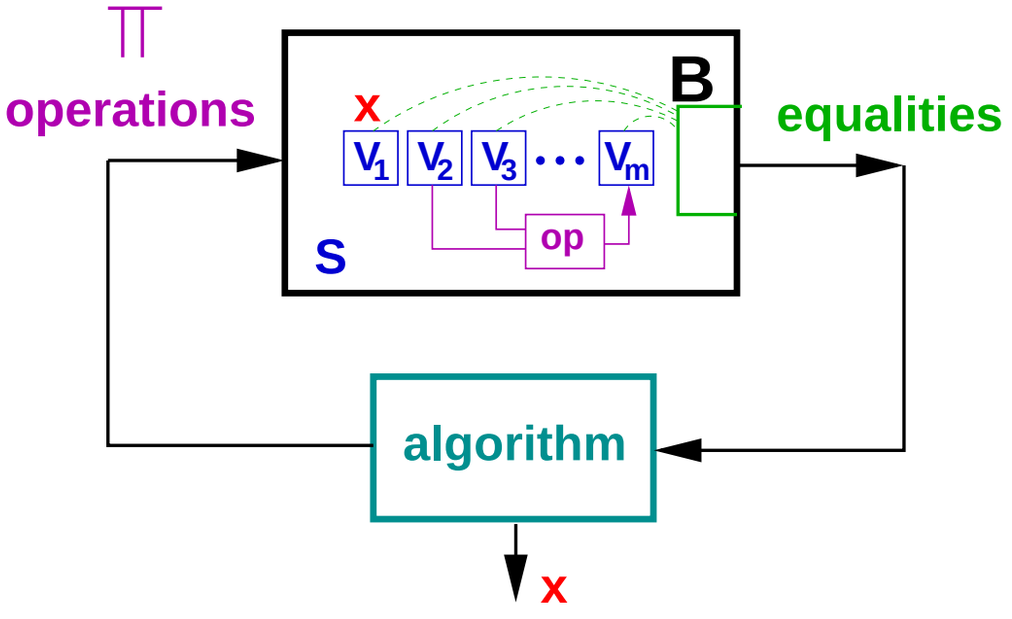
ETH Zurich, www.crypto.ethz.ch

Number Theory and Computational Cryptography
Sept. 29 - Oct. 2, 2010, Warsaw.

Diffie-Hellman protocol: mechanical analog



Computing DL, using CDH oracle



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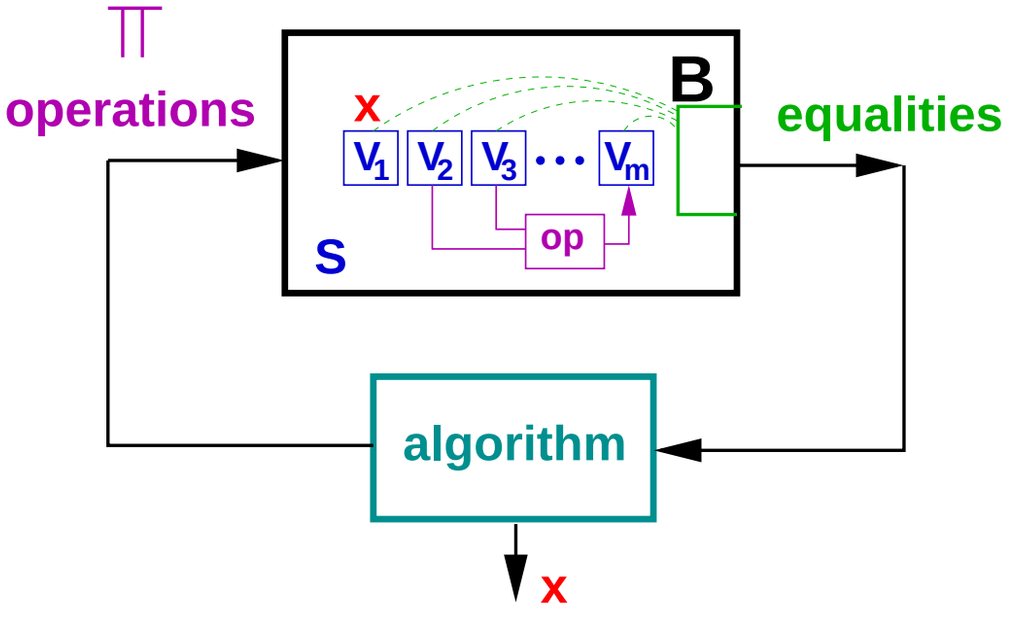
Computing DL, using CDH oracle



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Theorem: For (almost) every n, if $\Pi = \{1, +, \cdot\}$ there exists a polynomial-time extraction algorithm.

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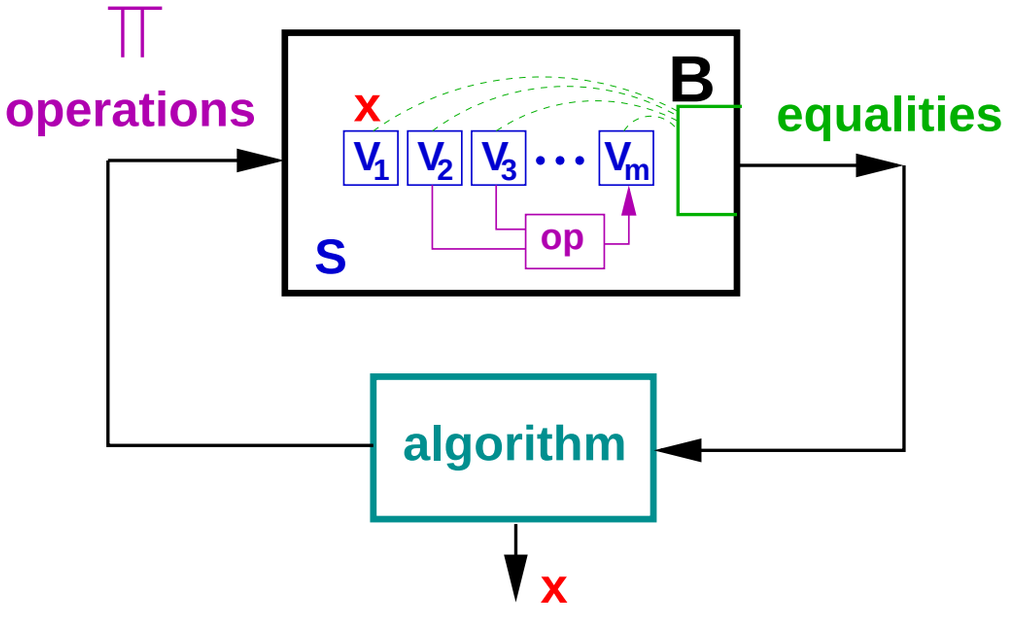

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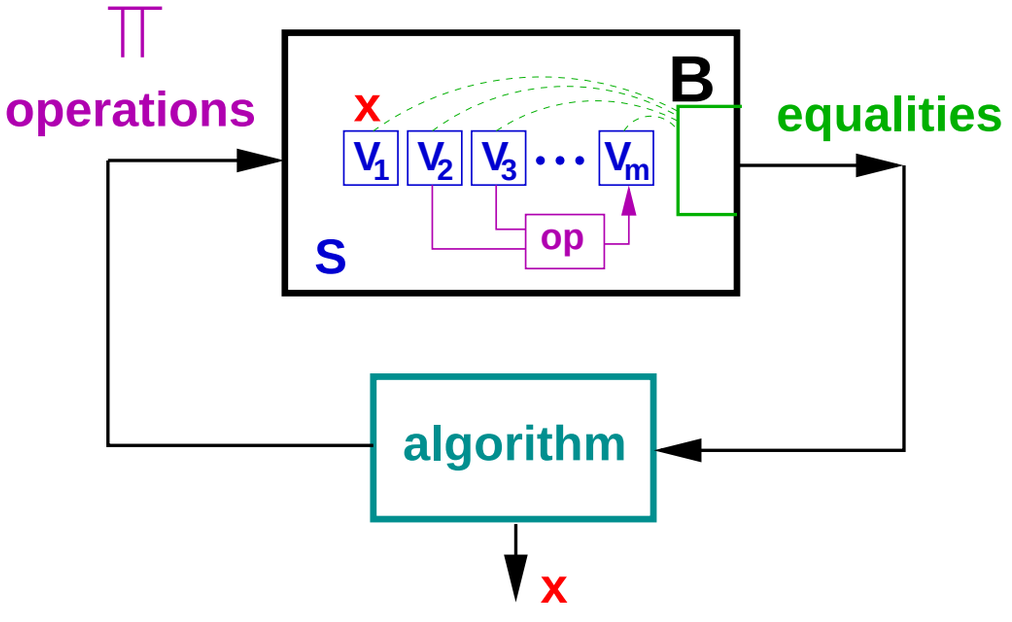

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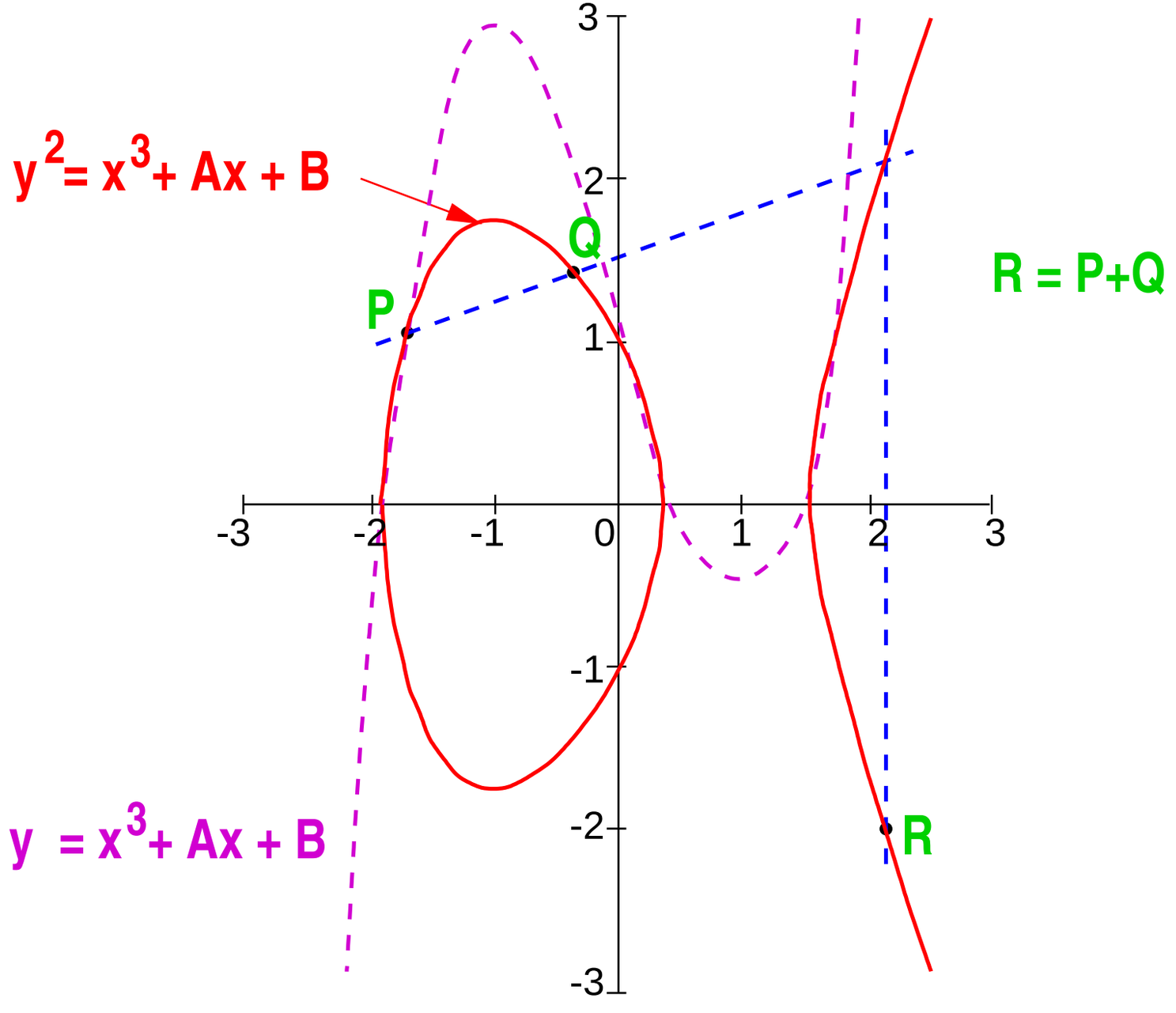
CDH–DL equivalence: General DH-groups for general order n can be designed together with the reduction algorithm.

Elliptic curve over the field R



Elliptic curves

An elliptic curve with parameters A and B over a given field $\mathbf{F}$ (with $\mathrm{char}(\mathbf{F}) \neq 2, 3$), where $4A^3 + 27B^2 \neq 0$, is the set of points:

$$E_{A,B} = \{(x, y) : \ y^2 = x^3 + Ax + B\} \cup \{\infty\}.$$

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 \neq x_2$. Then $R = P + Q = (x_3, y_3)$, where:

$$\lambda = \begin{cases} (y_2 - y_1)/(x_2 - x_1) & \text{if } x_1 \neq x_2 \\ (3x_1^2 + A)/(2y_1) & \text{if } x_1 = x_2,\ y_1 = y_2. \end{cases}$$

$$\begin{aligned} x_3 &= \lambda^2 - x_1 - x_2 \\ y_3 &= \lambda(x_1 - x_3) - y_1 \end{aligned}$$

Facts about elliptic curves over finite fields

- (Hasse, 1934). The order of the elliptic curve $E_{A,B}$ over $GF(q)$ lies in the interval

$$[q + 1 - 2\sqrt{q},\ q + 1 + 2\sqrt{q}].$$

- (Waterhouse, 1969). If q is prime, every order in this interval is assumed for some parameters A and B.

- (Rück, 1987). This is true even when we require the curve to be cyclic.

- (Lenstra, 1987). In the interval $[q + 1 - \sqrt{q},\ q + 1 + \sqrt{q}]$, the orders of elliptic curves are close to evenly distributed.

- For most elliptic curves (except e.g. super-singular curves), no DL-algorithm faster than the generic algorithms is known.

DH-DL equivalence [M94]

Let q be a divisor of n. We consider computing x modulo q.

Equality mod q: $x_i \equiv x_j \pmod{q}$ iff $(n/q)x_i = (n/q)x_j$

Inverses mod q: $x_i^{q-2} \equiv x_i^{-1} \pmod{q}$

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Square roots mod q: There exist algebraic square root algorithms. For instance, if $q \equiv 3 \pmod{4}$, then $x_i^{(q+1)/4}$ is a square root of x_i (provided x_i is a square mod q).

Auxiliary group technique:

1. Embed x in (the coordinate of) an element a_x of a finite **auxiliary group** H defined algebraically over $GF(q)$.

   In other words, $a_x = (x_i, x_j, \ldots, x, \ldots, x_k)$ for some $i, j, k, \ldots$.

2. Use the black-box's equality info. to compute a_x explicitly.

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Implementation of Step 2:

- Compute the discrete logarithm z of a_x in H with respect to a generator h of H (if H is cyclic), using a generic algorithm (Pohlig-Hellman): $h^z = a_x$ in H

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Note: This is efficient only if the order of H is smooth.

Example: cyclic elliptic curve

Theorem (Rück, 1987)**:** For every $d \in [q - 2\sqrt{q} + 1, q + 2\sqrt{q} + 1]$, there exists a cyclic elliptic curve over $GF(q)$ of order d.

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Theorem (Rück, 1987)**:** For every $d \in [q-2\sqrt{q}+1, q+2\sqrt{q}+1]$, there exists a cyclic elliptic curve over $GF(q)$ of order d.

1. Choose a smooth $d \in [q-2\sqrt{q}+1, q+2\sqrt{q}+1]$.
   (Such a d exists under a plausible number-theoretic conjecture.)
2. Find the parameters $A, B \in GF(q)$ of a cyclic elliptic curve $E_{A,B}(q)$ with order d, together with a generator P.
3. Test $x, x+1, \ldots$ for quadratic residuosity mod q until $x+e$ is a quadratic residue $((x+e/q) = 1)$.
4. Compute y such that $y^2 \equiv x^3 + Ax + B \pmod{q}$.
5. Using the Pohlig-Hellman generic algorithm, compute the DL z of the point $(x+e, y)$ to the base P in $E_{A,B}(q)$.
   This is efficient because $|E_{A,B}(q)| = d$ is smooth.
6. Compute $(x', y) := z \cdot P$ and $x := x' - e$.

A number-theoretic conjecture

Let $p_1(n)$ denote the largest prime factor of n, and let

$$\nu(n,t) := \min_{n-t \le d \le n+t} p_1(d).$$

Let $\psi(n,m)$ be the number of integers in $[1,n]$ that contain no prime factor greater than m.

Theorem (Canfield, Erdös, Pomerance, 1983)**:** For any fixed u,

$$\psi(n, n^{1/u}) = \frac{n}{u^{(1+o(u))u}}.$$

Example: $\lim_{n\to\infty} \frac{\psi(n,n^{1/2})}{n} = 1 - \ln 2 = 0.31.$

Heuristic reasoning:

- Assumption: This also holds if u is a moderately growing function of n.
- If for some b the fraction $\psi(n,b)/n$ of b-smooth integers $\le n$ is substantially greater than $1/t$, then it is highly probable that there exists at least one b-smooth integer in the interval $[n-t, n+t]$, i.e., $\nu(n,t) \le b$.

Let $t = n^{1/s}$ and $b = \log^k n$. Setting $b = n^{1/u}$ we obtain $u = \log n/(k \log\log n)$ and hence

$$\psi(n, \log^k n)/n \;\approx\; u^{-u} \;=\; \left(\frac{\log n}{k \log\log n}\right)^{-\frac{\log n}{k \log\log n}} \;\gg\; n^{-1/k}$$

because

$$\log(u^u) = \frac{\log n}{k \log\log n}(\log\log n - \log\log\log n - \log k) < \frac{\log n}{k}.$$

For $k > s$ there exist on the order of $n^{1/s-1/k} \gg 1$ $(\log^k n)$-smooth integers in the interval $[n - n^{1/s}, n + n^{1/s}]$. (We need the case $s = 2$.)

Conjecture: $\nu(n, n^{1/s}) = (\log n)^{(1+o(s))s}$.

Stronger form: $\nu(n, n^{1/s}) = (\log n)^{s+o(s)}$.

Weaker, but still sufficient forms: $\nu(n, n^{1/s}) = (\log n)^{s^{O(1)}}$ or, for some function f: $\nu(n, n^{1/s}) = (\log n)^{f(s)}$.

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Construction of G of third type:

1. Choose a large smooth number m.
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Alternative: Use (Lay, Zimmer, 1994) to construct an elliptic curve of given order m together with the field $GF(q)$ over which it is defined.

Breaking RSA Generically is Equivalent to Factoring

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ETH Zurich, www.crypto.ethz.ch

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