Breaking RSA Generically is Equivalent to Factoring

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April 27, 2009

RSA public key cryptosystem

Public Key: (n, e) $\quad n = pq \quad$ $\gcd(e, \phi(n)) = 1$.

Encryption: $m \mapsto c = m^e \mod n$.

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Is it necessary to factor n to break RSA?

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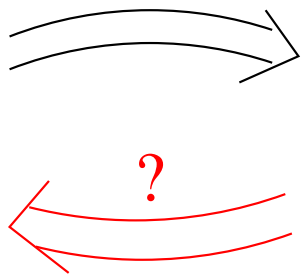
Factoring n  Breaking RSA

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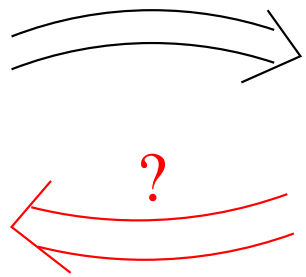
Factoring n  Breaking RSA

- General models of computation: Unknown
- The generic model of computation: This paper!!

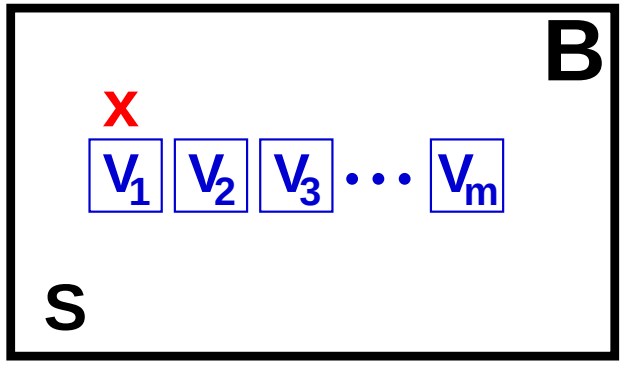
The generic model of computation

Generic Algorithms:

- Introduced by Shoup in ’97
- Representation independent
- Used to prove lower bounds
  (DL, CDH, DDH, etc)

The generic model of computation

The model [Maurer05]



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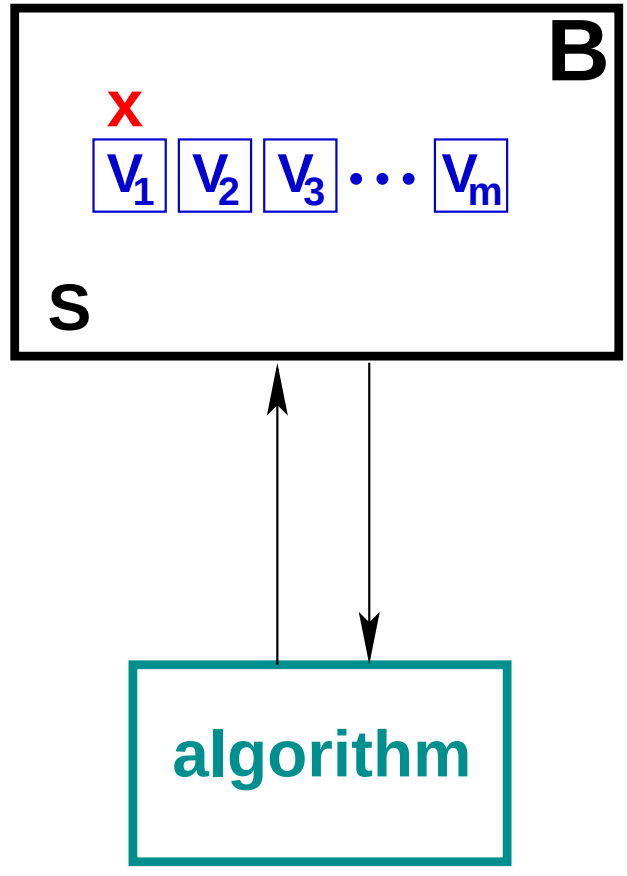

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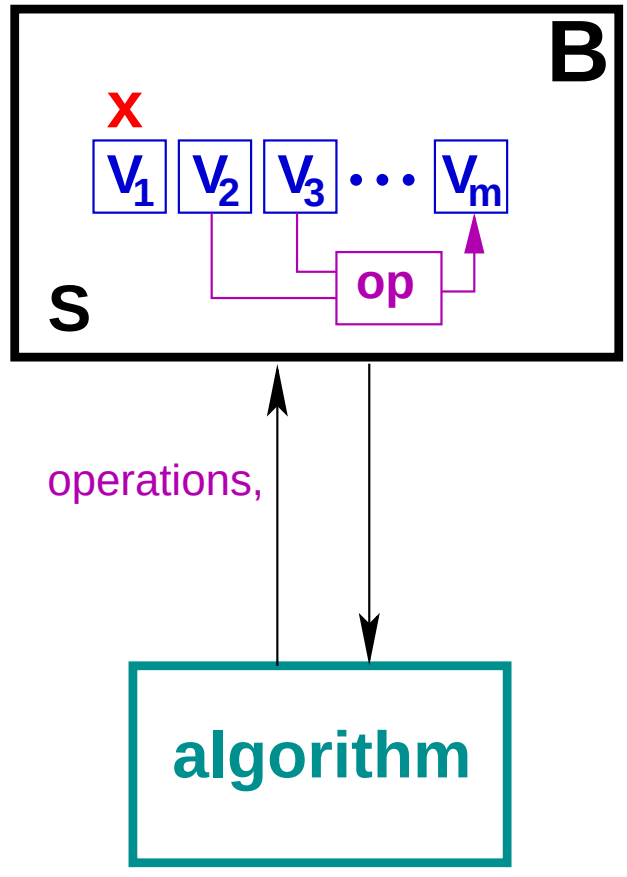

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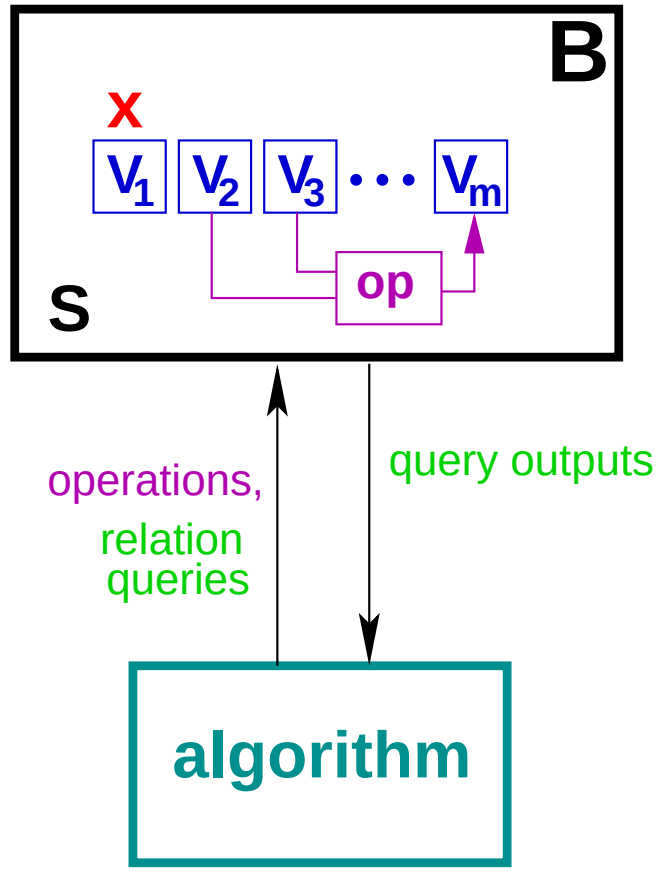

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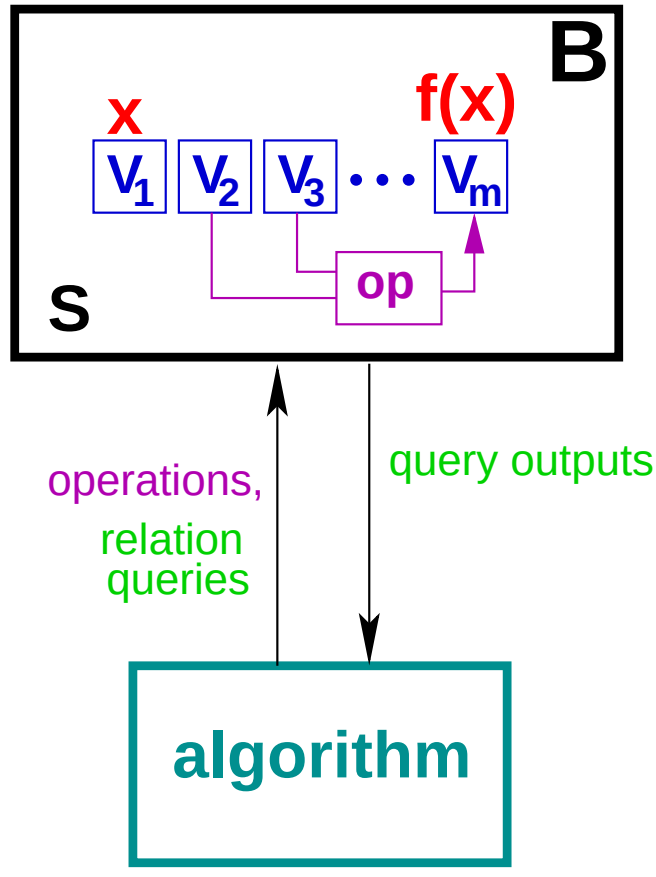
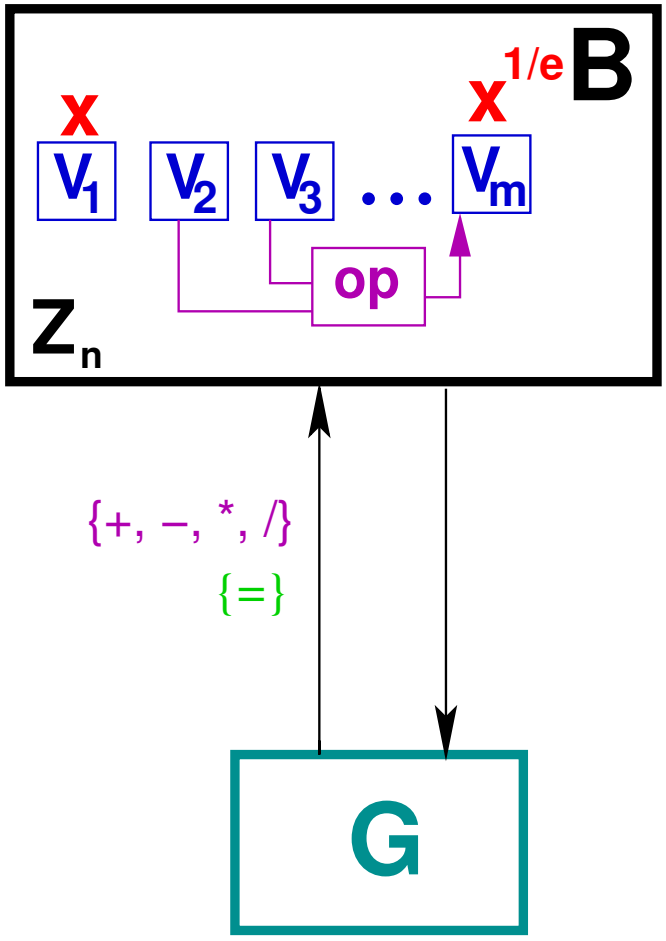


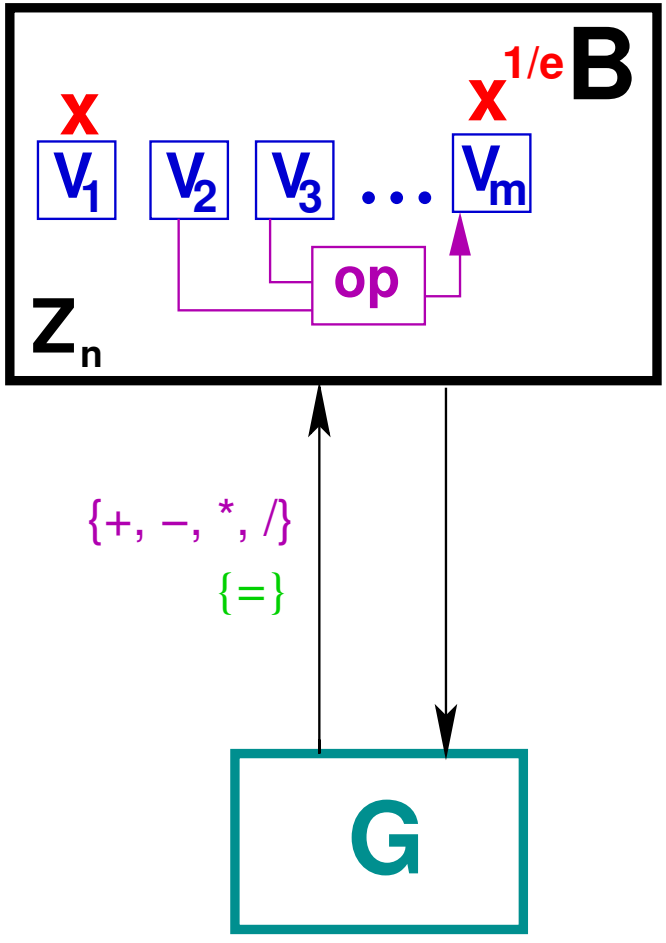
Generic ring algorithms(GRAs) to compute e-th roots

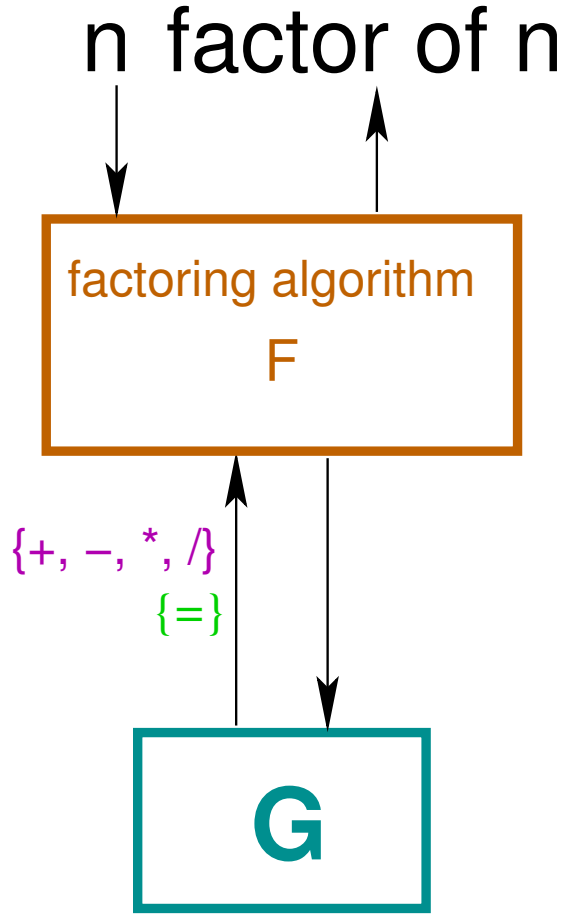
- Only allowed $\{+, -, \cdot, /\}$, $\{=\}$
- Task of G: Compute $x^{1/e}$ modulo n.



The main result

$G : x \mapsto x^{1/e} \quad \Longrightarrow \quad F^G : n \mapsto$ factor of n





Comparison to previous results

	operations	queries	GRAs	e	Result
DK02	$\{\cdot\}$	$\{=\}$	-	any	e-th root hard
Brown06	$\{+,-,\cdot\}$	$\{\}$	deterministic	small	e-th root $\Rightarrow$ Factoring
LR06	$\{+,-,\cdot\}$	$\{=\}$	deterministic	small	e-th root $\Rightarrow$ Factoring
This paper	$\{+,-,\cdot,/\}$	$\{=\}$	**randomized**	**any**	e-th root $\Rightarrow$ Factoring

Outline

- Straight line programs (SLPs): $\{+, -, \cdot, /\}$, $\{\}$.
- Generic ring algorithms (GRAs): $\{+, -, \cdot, /\}$, $\{=\}$.

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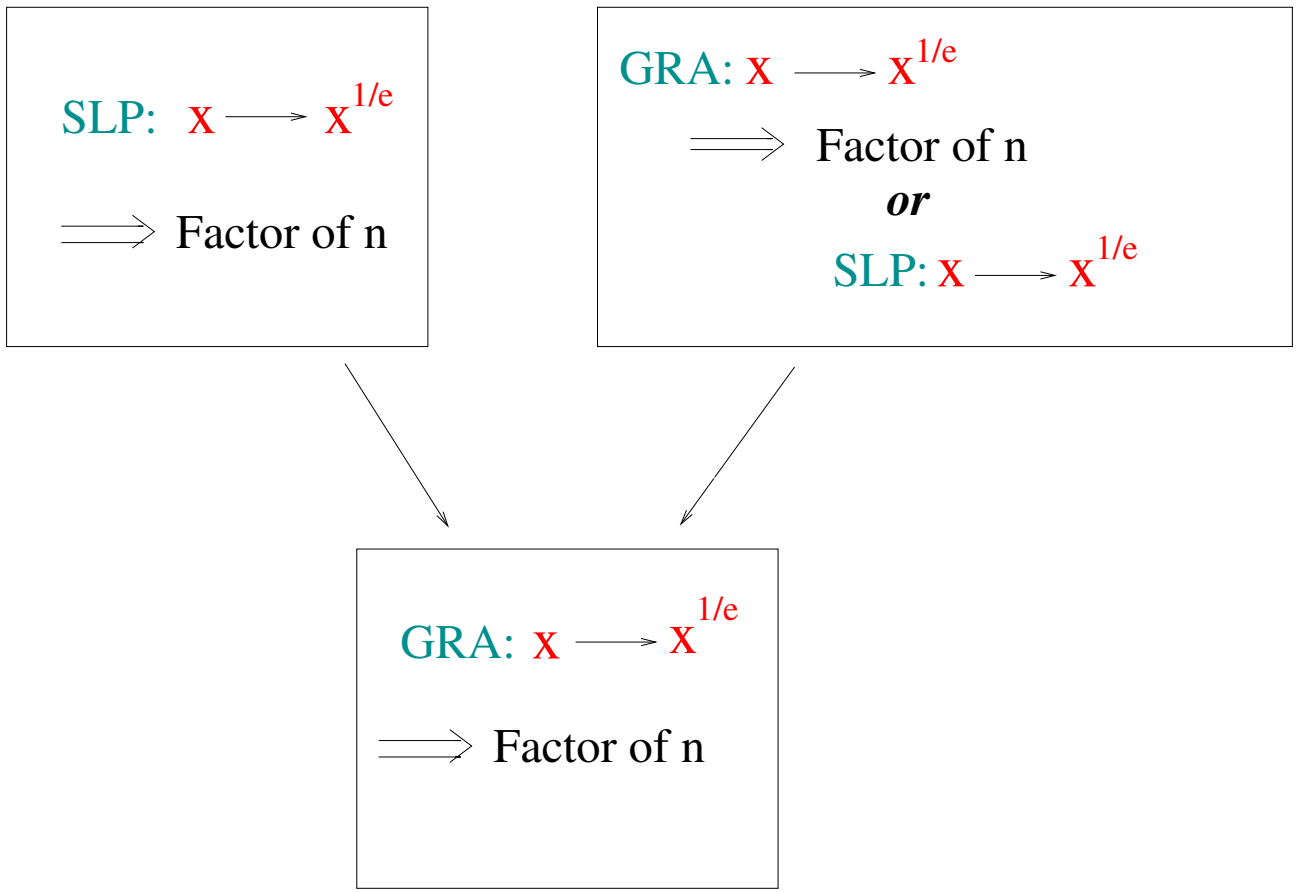

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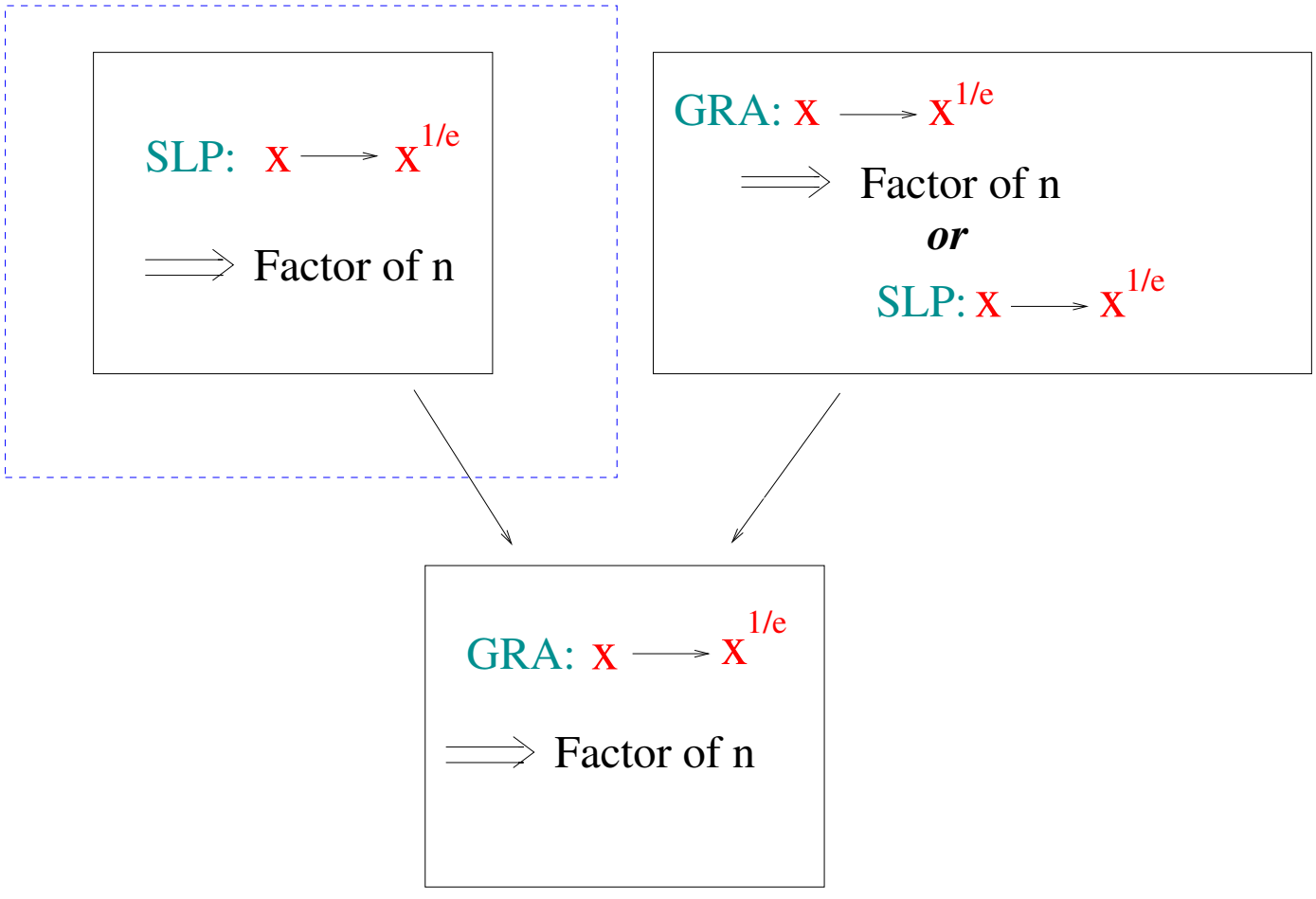

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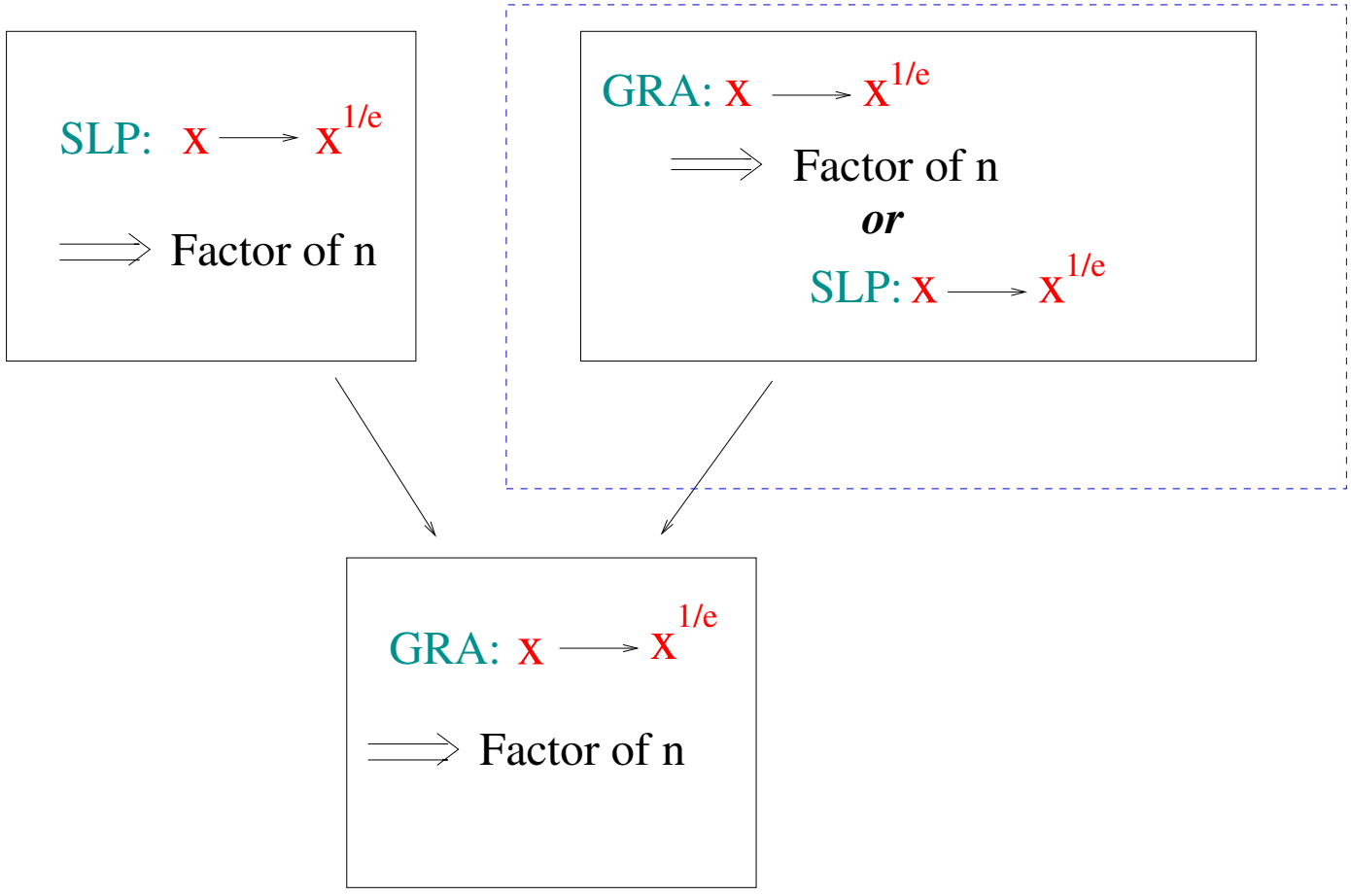

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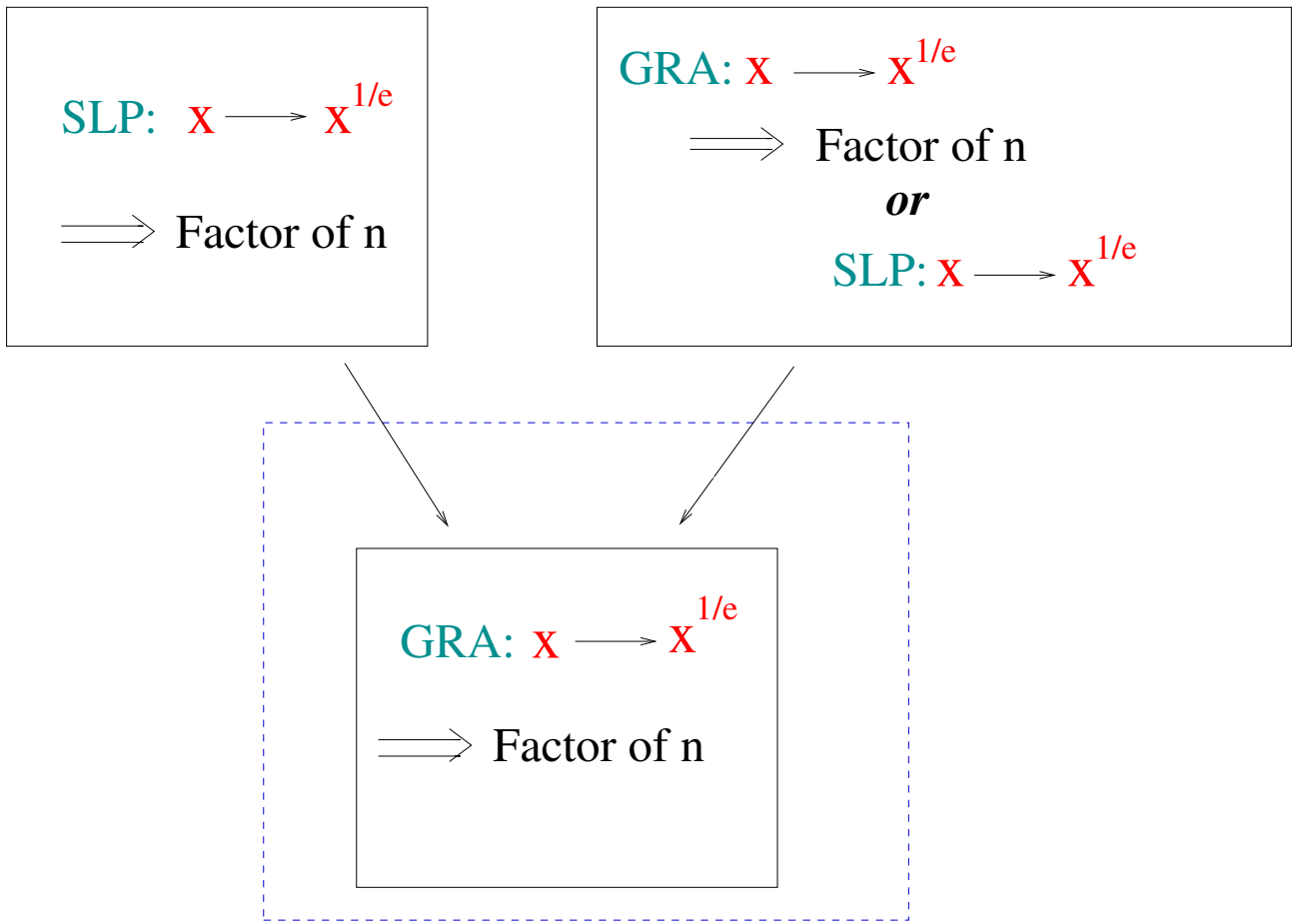

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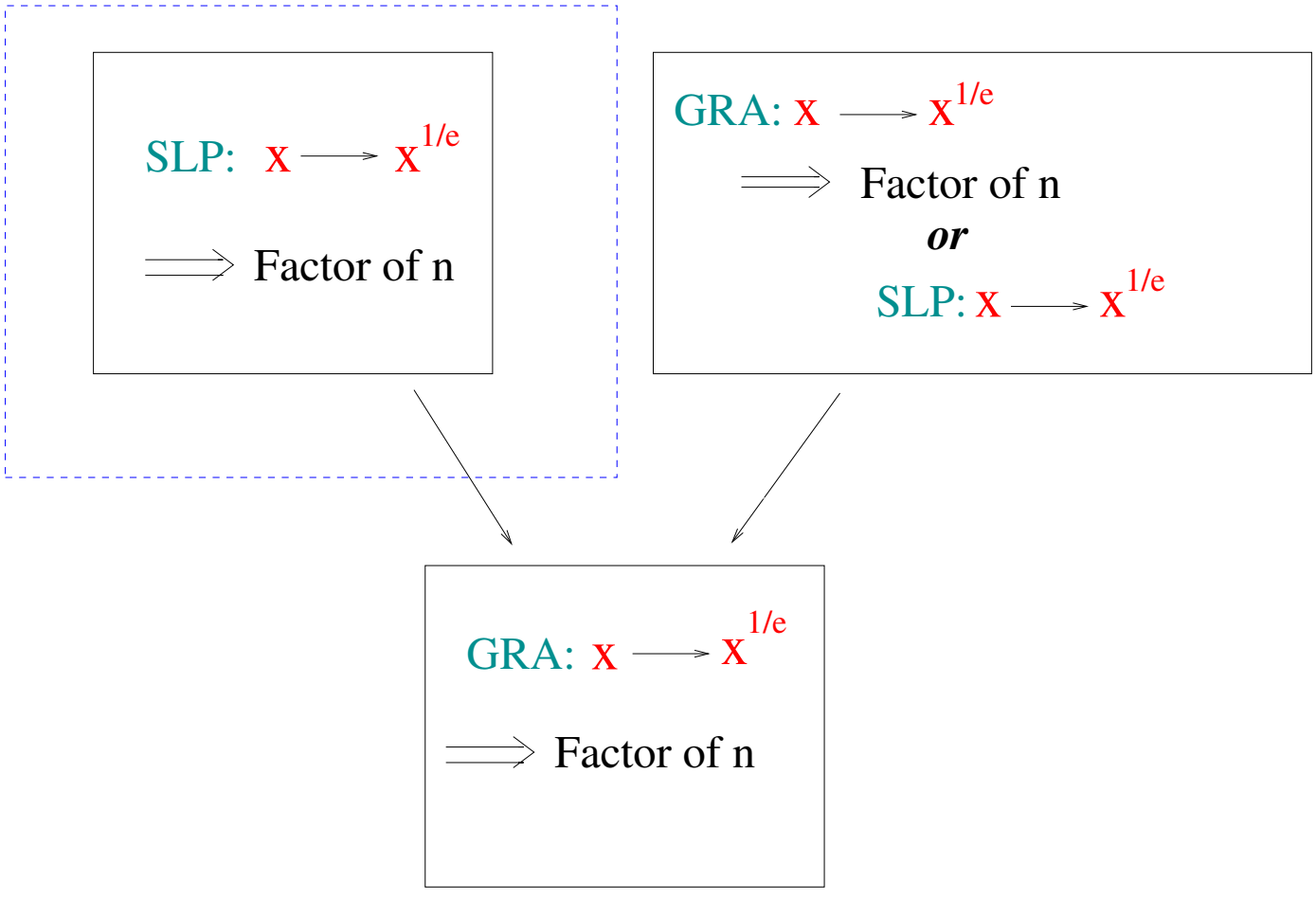

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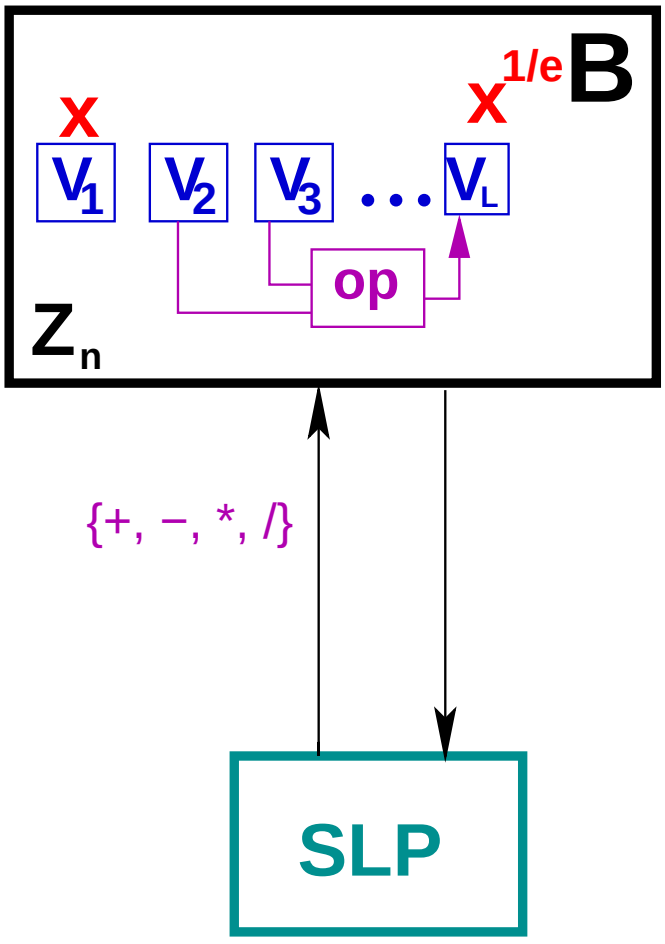

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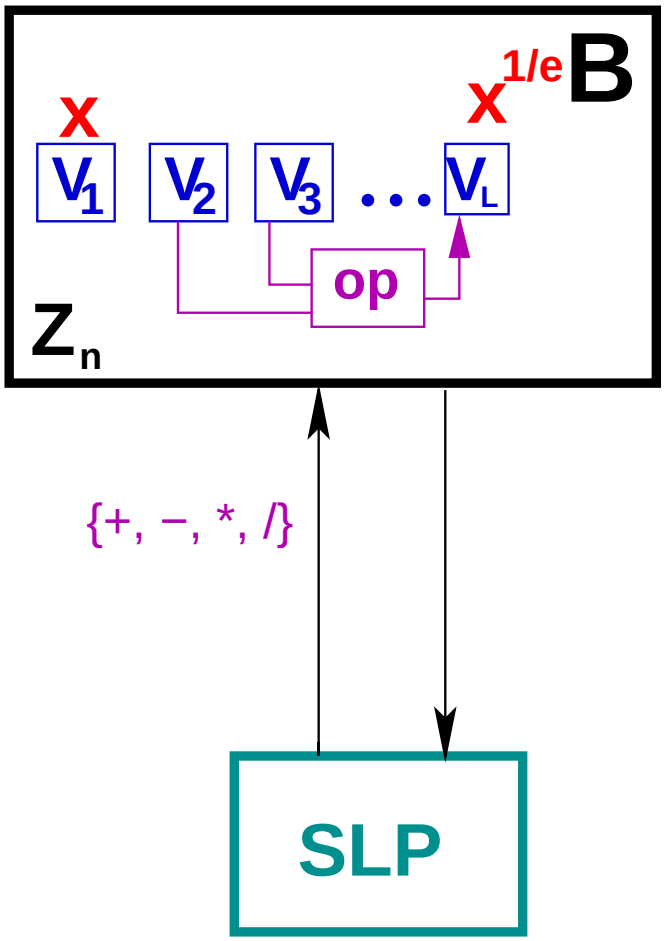

SLP of length L: Sequence of triples $(i_2, j_2, \circ_2), \ldots, (i_L, j_L, \circ_L)$

- $\circ_k \in \{+, -, \cdot, /\}$.

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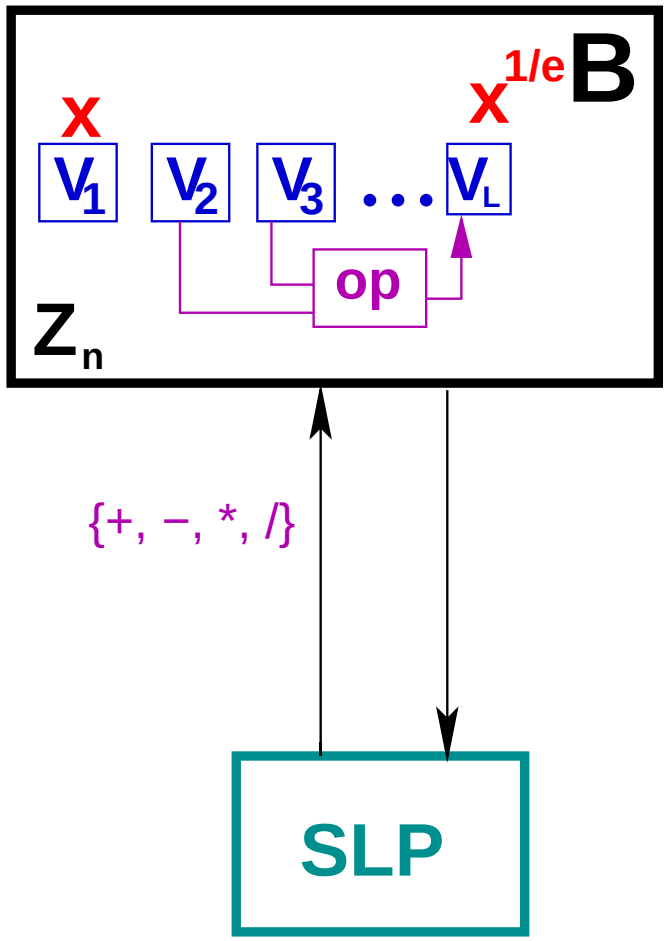

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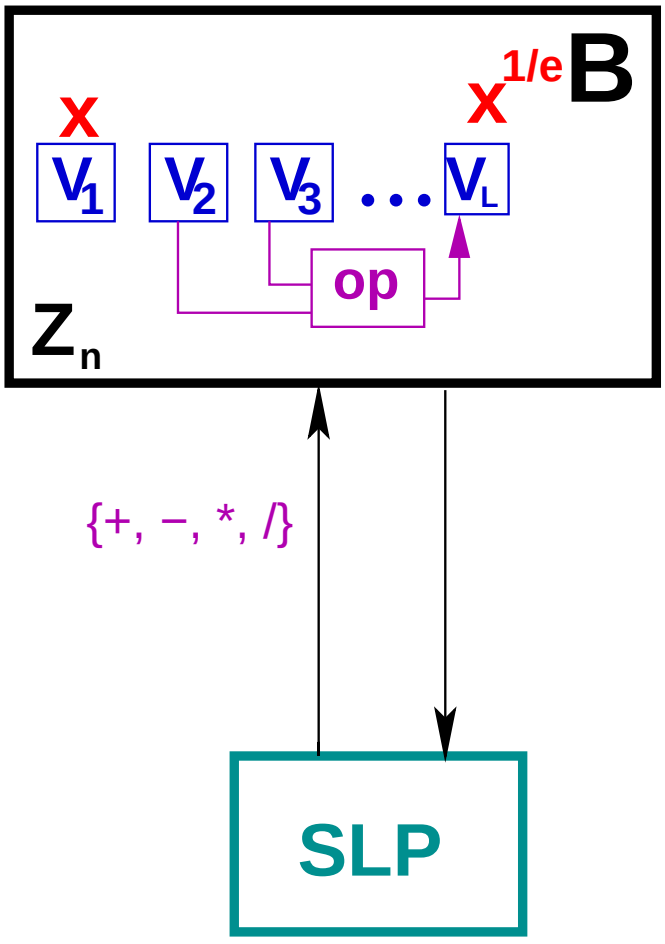

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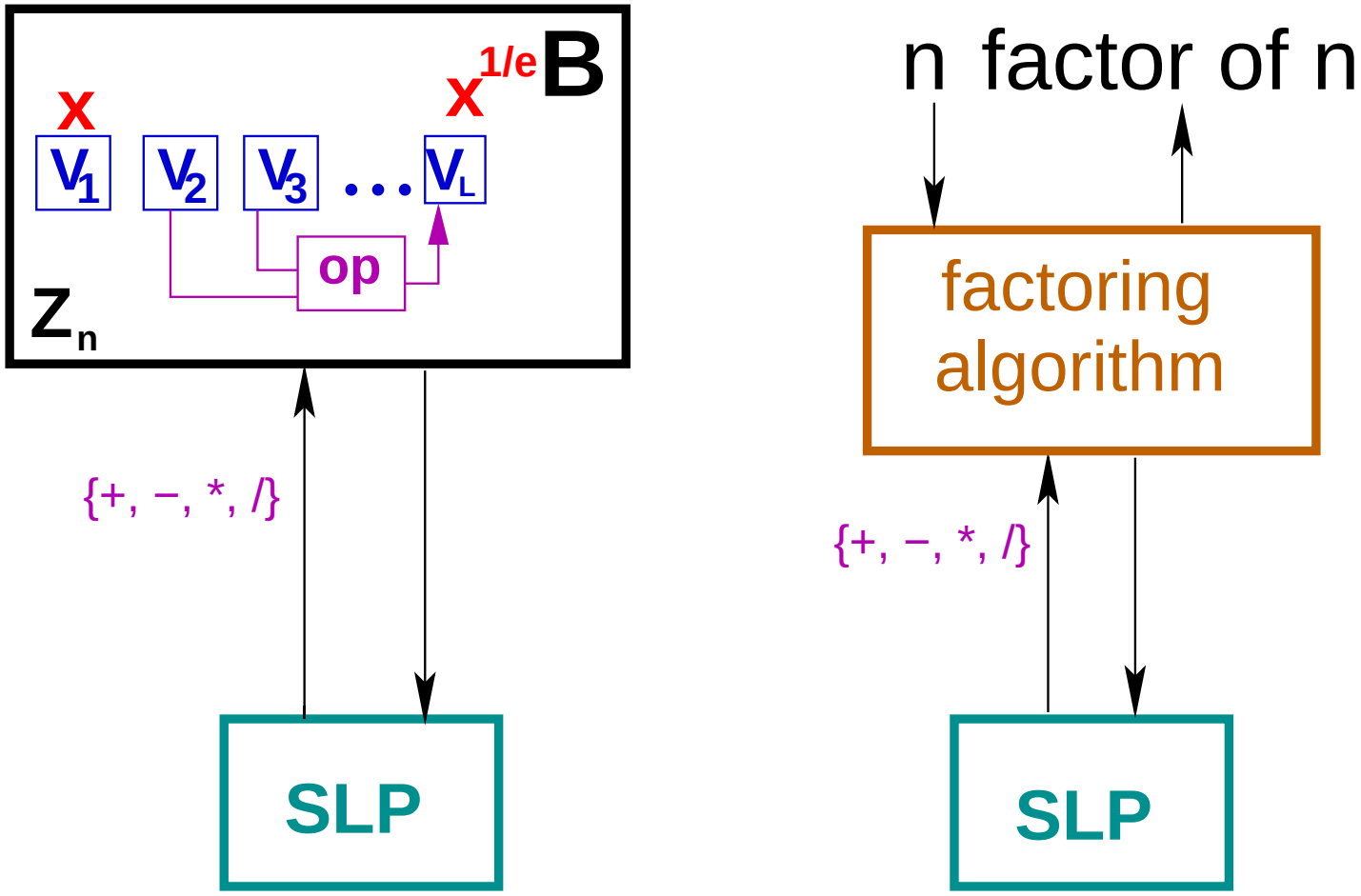
- $V_k = V_{i_k} \circ_k V_{j_k} = \frac{P_k(x)}{Q_k(x)}$.

SLP for computing e-th roots $\Rightarrow$ Factoring

$SLP : x \mapsto x^{1/e} \quad \Longrightarrow \quad F_1{}^{SLP} : n \mapsto$ factor of n



More formally...

Lemma

$\exists$ *efficient* F_1, *given an L-step SLP* $S : x \mapsto \frac{P(x)}{Q(x)}$ *s.t.,*

$$\Pr_{x \in_R \mathbb{Z}_n} \left(\left(\frac{P(x)}{Q(x)} \right)^e \equiv_n x \right) \geq \mu$$

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A few facts:

- $\deg(P(x)^e - x \cdot Q(x)^e) \quad \leq \quad e \cdot 2^L + 1$
- $\exists$ $(4L + 4 \log e + 2)$-step SLP

$$S_1 \ : \ x \mapsto P(x)^e - x \cdot Q(x)^e.$$

Factoring algorithm

Let $f(x) = P(x)^e - x \cdot Q(x)^e$ $\qquad$ $S_1 \ : \ x \mapsto f(x)$.

The Algorithm

1. Choose random $h(x) \in \mathbb{Z}_n[x]$ of degree $d = L + log(e)$.
2. Choose random $r(x) \in \mathbb{Z}_n[x]/h(x)$.
3. Compute $z(x) = f(r(x))$ in $\mathbb{Z}_n[x]/h(x)$.
4. (i) Run Euclid$\big(h(x), z(x)\big)$ in $\mathbb{Z}_n[x]$.
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P

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$\bar{d}$.

Observation

- On a non-Euclidean domain,
  Euclid's algorithm fails $\Rightarrow$ it gives a zero-divisor.
- $\deg\Big(gcd_p\big(h(x), z(x)\big)\Big) \neq \deg\Big(gcd_q\big(h(x), z(x)\big)\Big)$
  $\Rightarrow$ Euclid$\big(h(x), z(x)\big)$ fails.

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$$\Longrightarrow \Pr\Big((x-s) \mid gcd_p\big(h(x), z(x)\big) \ \wedge \ gcd_q\big(h(x), z(x)\big) = 1\Big)$$
$$\geq \frac{1}{4d} \cdot \mu \cdot \frac{1}{2} = \frac{\mu}{8(L + \log e)}$$

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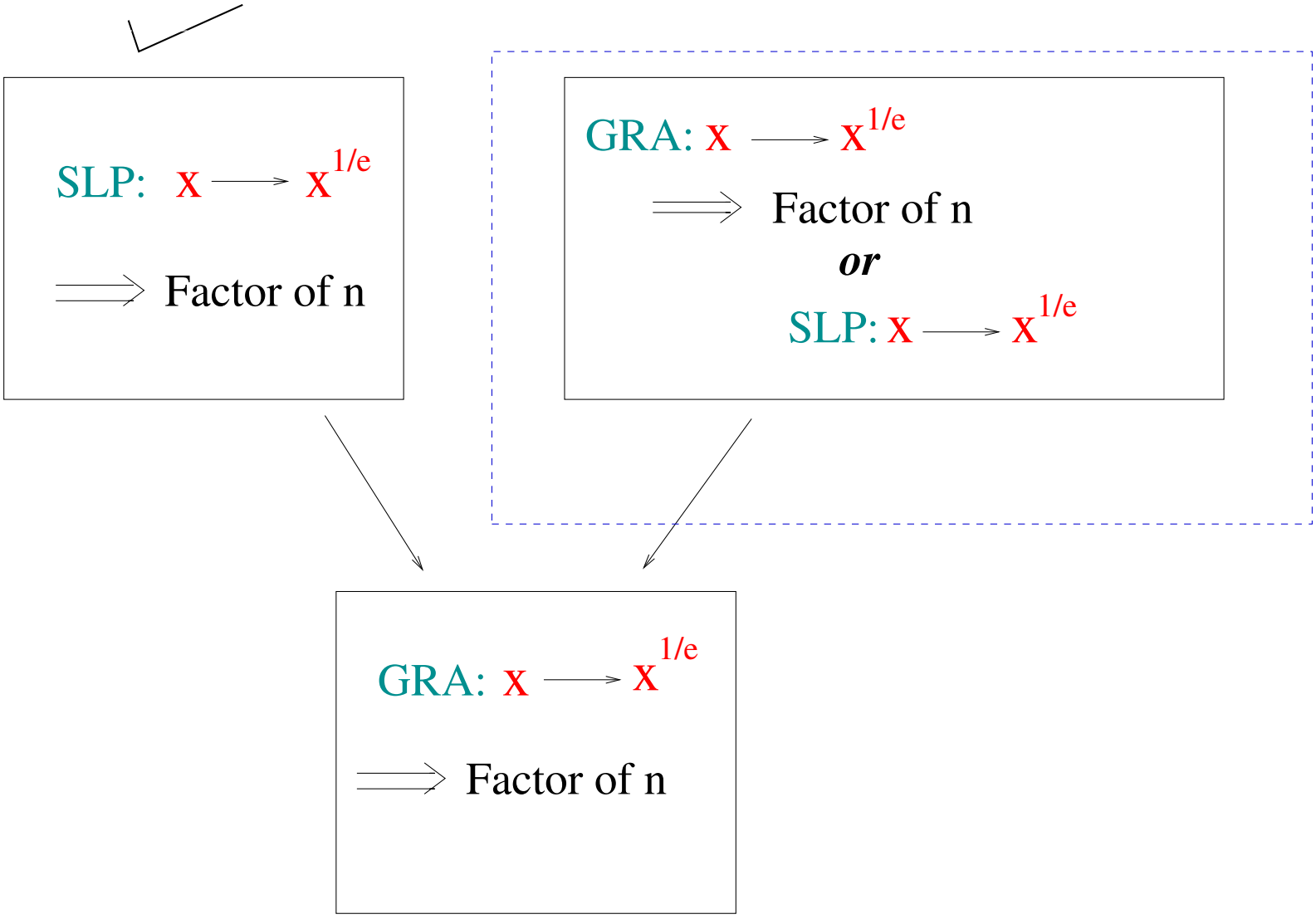
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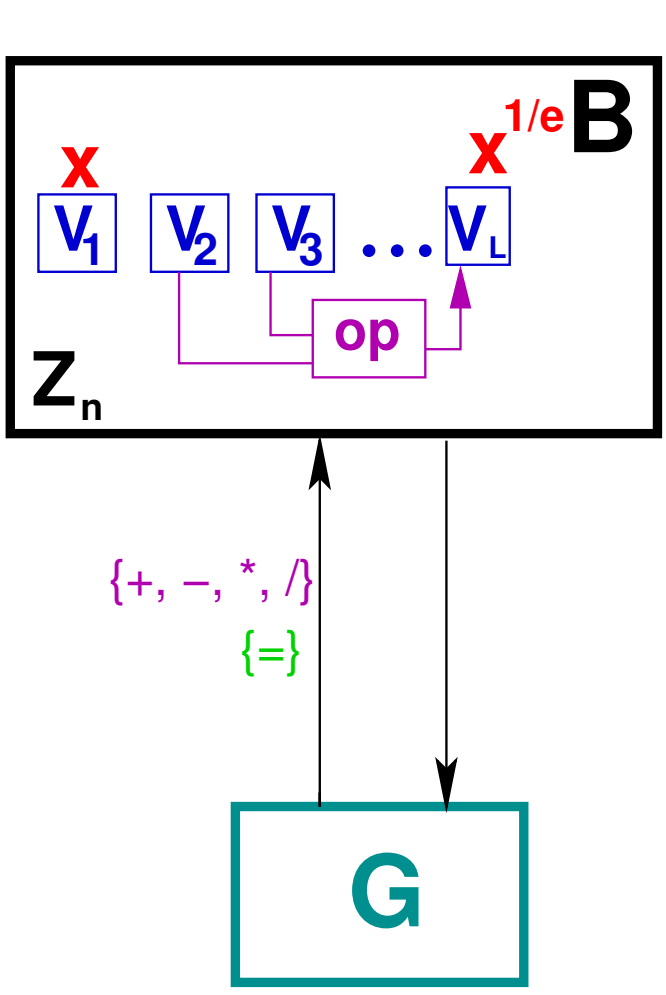
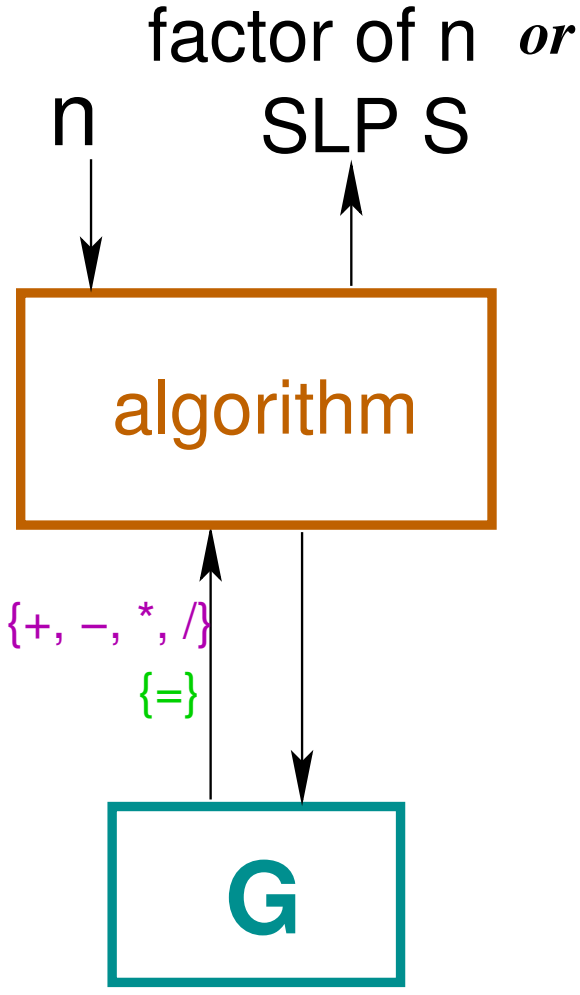
From SLPs to deterministic GRAs

Lemma

$G : x \mapsto x^{1/e} \implies A^G : n \mapsto$ *factor of n or S*





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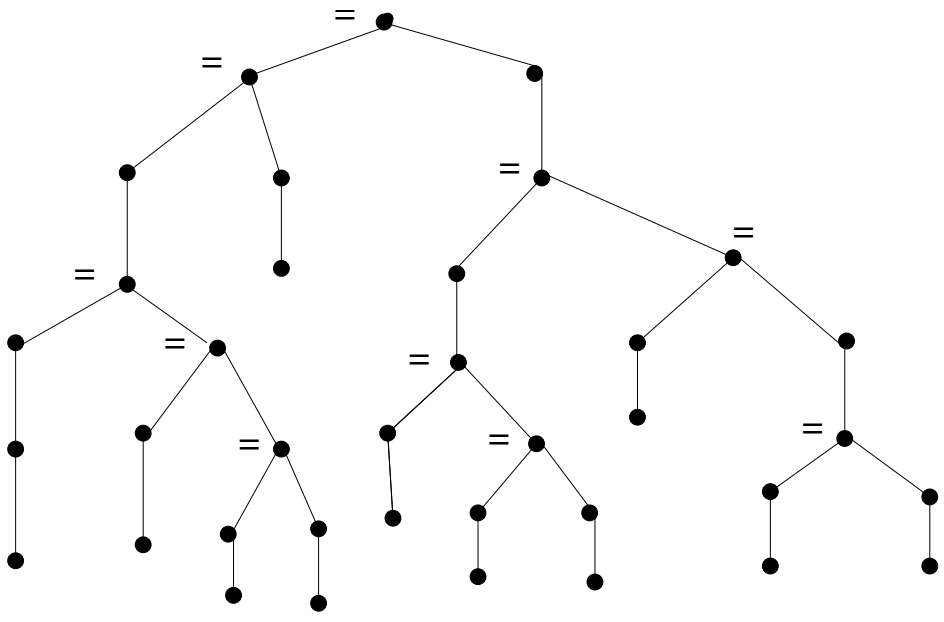
Lemma

$\forall \ f(x) \in \mathbb{Z}_n[x], \qquad \delta \leq \nu_n(f) \leq 1 - \delta \implies \eta_n(f) \geq \delta^{\frac{3}{2}}.$

From SLPs to deterministic GRAs: A proof sketch

GRA can be seen as a binary tree!

Every path from the root to a leaf corresponds to an SLP.



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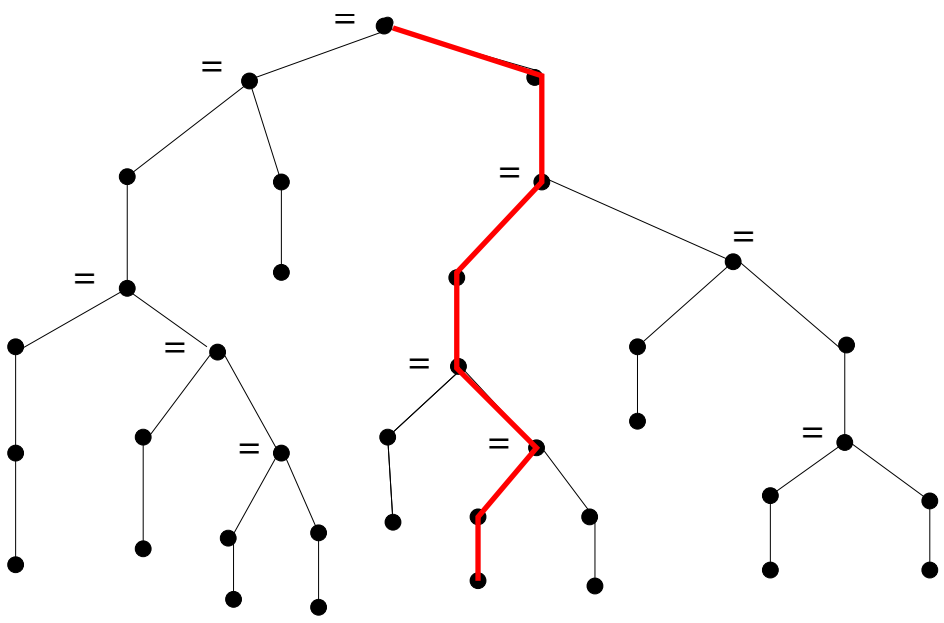

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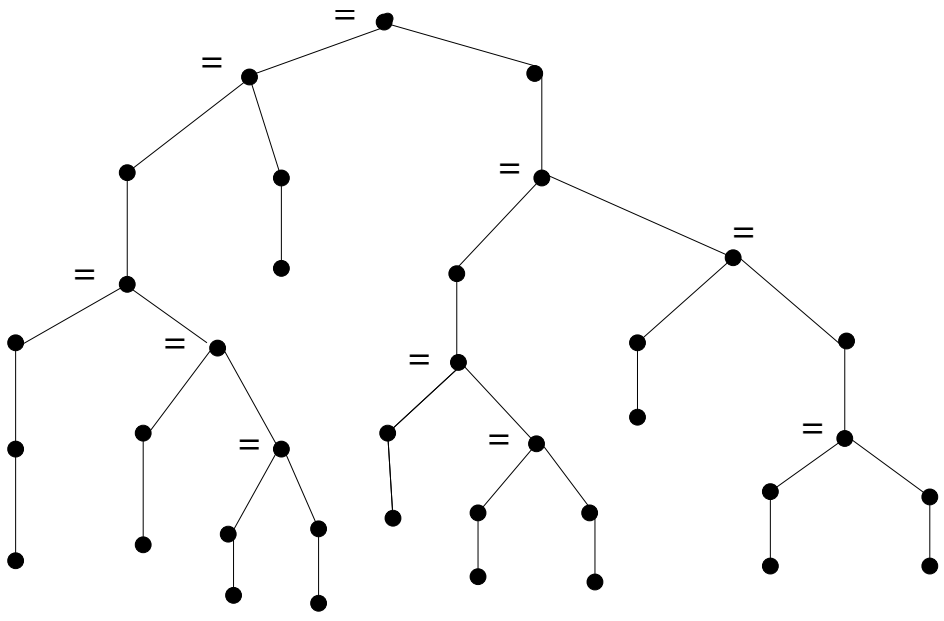

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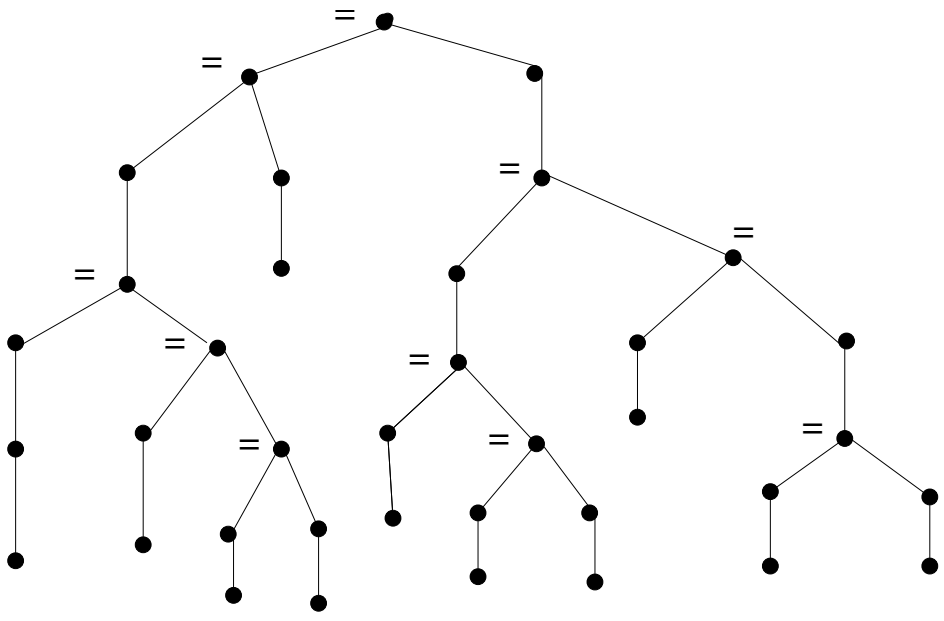

Equality query: $\frac{P_i}{Q_i} \stackrel{?}{=} \frac{P_j}{Q_j}$

Two cases:

- An equality query with $\delta \leq \nu_n(P_iQ_j - Q_iP_j) \leq 1-\delta$.

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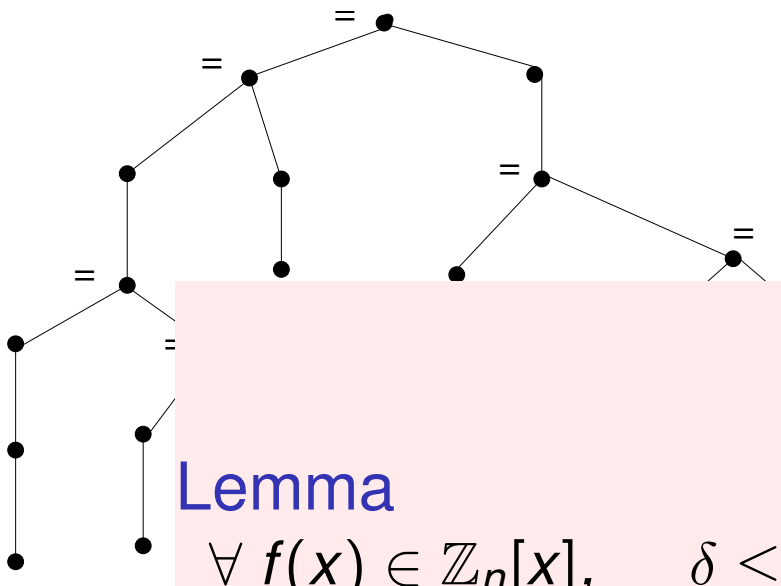

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Lemma

$\forall\, f(x) \in \mathbb{Z}_n[x], \quad \delta \leq \nu_n(f) \leq 1-\delta \implies$ *Factor of n.*

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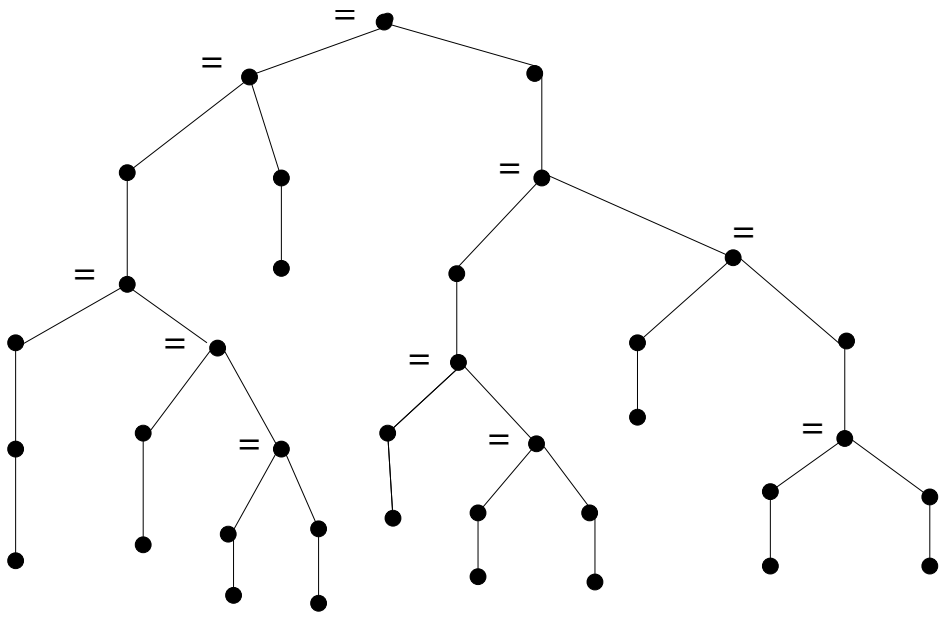

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Two cases:

- An equality query with $\delta \leq \nu_n(P_iQ_j - Q_iP_j) \leq 1-\delta$.
  - Can be used to factor n.

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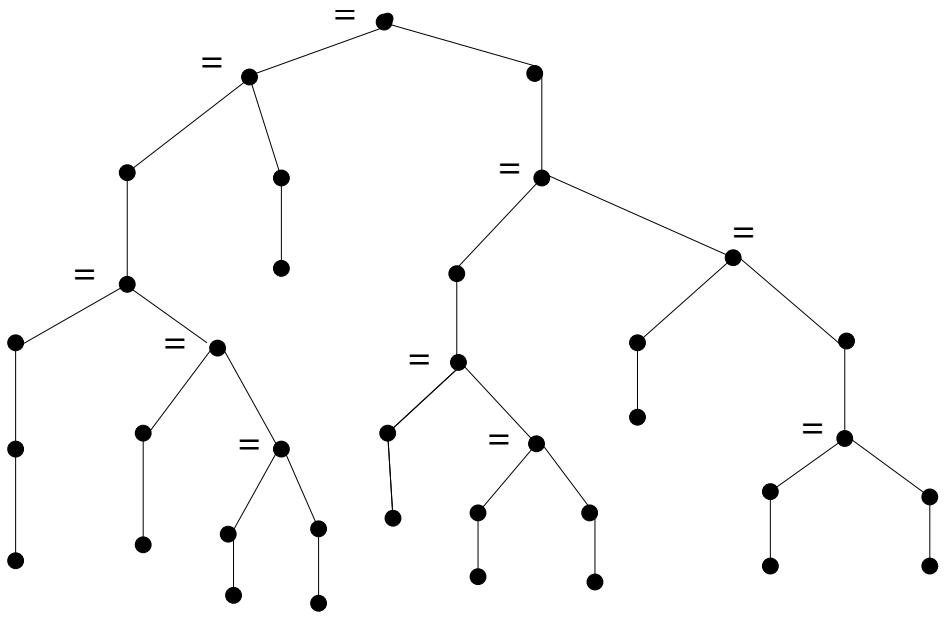

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- All equality queries have $\nu_n(P_iQ_j - Q_iP_j) > 1-\delta$ or $\nu_n(P_iQ_j - Q_iP_j) < \delta$.
  -

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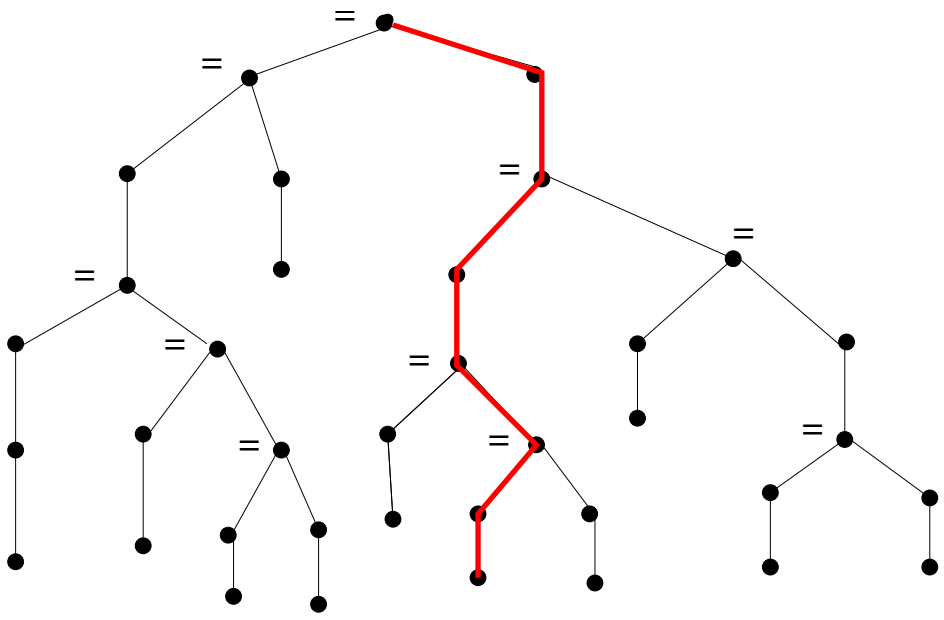

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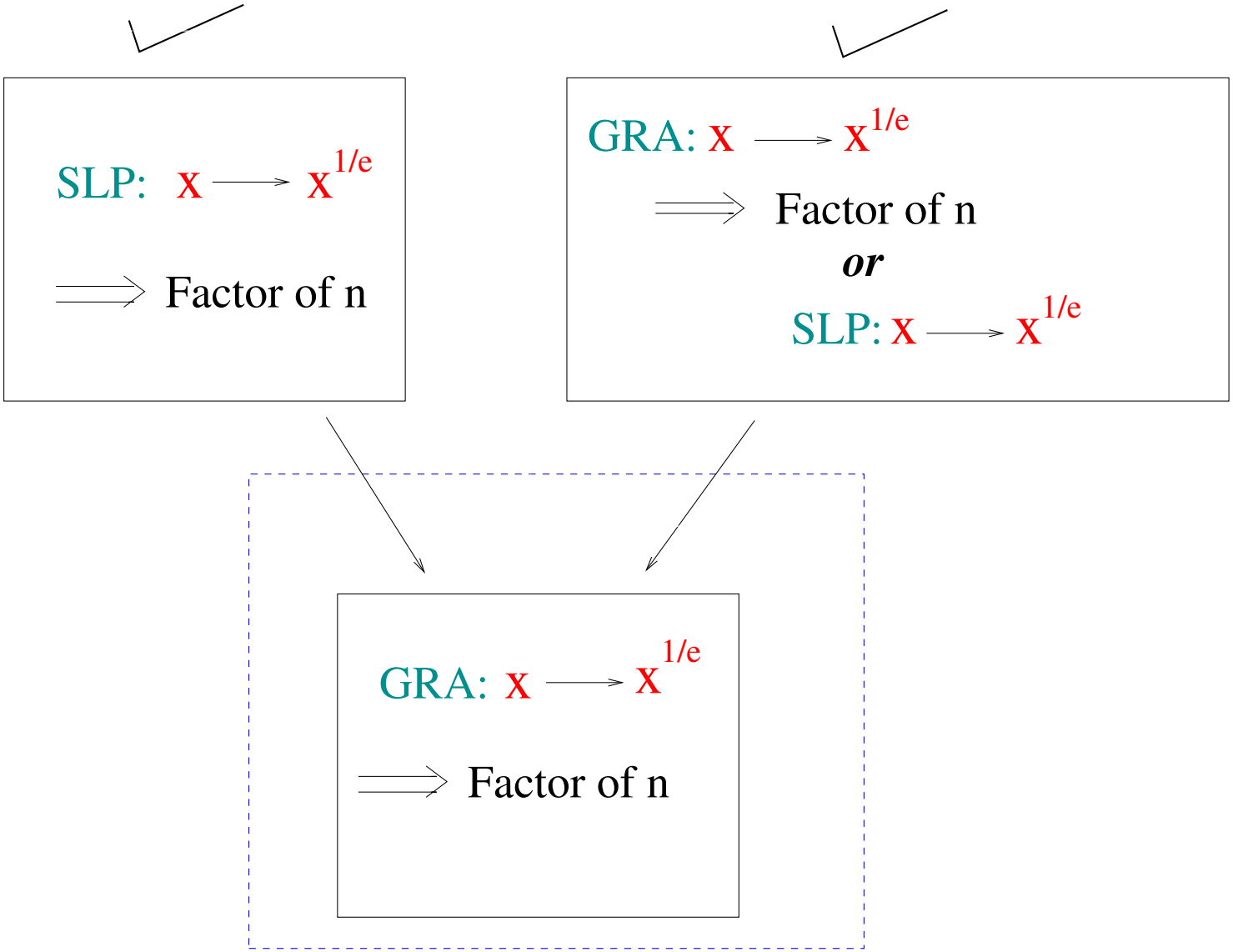
- All equality queries have $\nu_n(P_iQ_j - Q_iP_j) > 1-\delta$ or $\nu_n(P_iQ_j - Q_iP_j) < \delta$.
  - Gives an SLP.

Outline


SLP: $x \longrightarrow x^{1/e}$
$\Longrightarrow$ Factor of n
GRA: $x \longrightarrow x^{1/e}$
$\Longrightarrow$ Factor of n
or
SLP: $x \longrightarrow x^{1/e}$
GRA: $x \longrightarrow x^{1/e}$
$\Longrightarrow$ Factor of n

Generalize to randomized GRAs

Definition

A randomized GRA $\mathcal{G}$ is a GRA where the choice of the operation at each step is randomized.

Theorem

$\exists\ F$ *s.t.*, $\mathcal{G} : x \mapsto x^{1/e} \implies F^{\mathcal{G}} : n \mapsto$ *factor of* n

Proof: Less trivial than it may appear. Refer paper.

Conclusions

Breaking RSA $\Longrightarrow$ Factor n *or* Exploit bit representation.

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Breaking RSA $\Longrightarrow$ Factor n *or* Exploit bit representation.

New evidence: Breaking RSA may be equivalent to factoring.

Thank You